Response Spectra of Quasi-Stationary Random Excitations

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Abstract

One of the most important problems in the field of earthquake engineering is to suppose reasonable earthquake excitation patterns for the dynamic analysis of structures. In particular, in the response analysis of a structure for moderately intense earthquakes, it may be plausible to suppose a statistical model of earthquake excitations taking into account the seismicity and the dynamic characteristics of the ground at the site of the structure.

In this paper, as one of the basic studies related to such artificial earthquake excitations as are used in the dynamic aseismic design of structures, the statistical characteristics of the response spectra of a quasi-stationary random excitation, which is defined as the product of a deterministic time-function and an ergodic stationary random process, are discussed.

The expressions of the mean value and the upper and lower limits of the response spectra of the quasi-stationary random excitations are obtained as the products of the root of the maximum value of energy spectral density of a modified quasi-stationary random process, which is approximately equal to the maximum value of the root mean square of the envelope of the output responses of a single-degree-of-freedom, damped oscillator applied to the quasi-stationary random excitations with a finite duration time, and the relevant multiplication factors which are expressed in terms of the characteristic values of the Rayleigh distribution and the amplitude probability distribution of the maximum value of the normalized random variable associated with a pseudo-stationary envelope of the non-stationary output response of the oscillator.

The analytical expressions of the energy spectral density and power spectral density of the modified quasi-stationary random process are presented for a case where the envelope of the quasi-stationary random process is expressed as the product of an arbitrary deterministic time-function and a cutoff operator in time domain, while the spectral density of the ergodic stationary random process is given by the product of a rational function and a band-limiting operator. The iterative method of evaluating the maximum value of the energy spectral density of the modified quasi-stationary random process is also discussed.

On the other hand, the multiplication factors which give the mean value and the upper and lower limits of the response spectra, together with the above-mentioned maximum value of the energy spectral density, are determined semi-experimentally by means of the simulation method in a case where the envelope is given by a step-function multiplied by the cutoff operator and the spectral density is a rational function multiplied by a band-limiting operator.

1. Introduction

In the earthquake response analysis of a structure for moderately intense earthquakes, it is important to suppose a reasonable statistical model of the earthquake excitations by taking into consideration the seismicity and the dynamic characteristics of the ground at the site of the structure, as well as the measure of aseismic safety and the spectral characteristics of the structure to be designed.[1)~3)] Even though many statistical models of earthquake excitations for the dynamic analysis of structures have been proposed by various investigators,[4)~13)] the authors dare to deal with the relevant problems in this paper, for its importance in earthquake engineering, mainly from the basic aspect of finding the statistical characteristics of the response spectra of the quasi-stationary random excitation, which is defined as the product of a deterministic time-function and a stationary random process.[1),3),13),14)] Of course, to suppose a definite quasi-stationary random process as a model of earthquake excitations which is usable in the response analysis of a structure, it is necessary to determine reasonably, from a comprehensive point of view, the deterministic time-function which gives the envelope of the quasi-stationary random excitations, as well as the stationary random process which provides the statistical properties of the quasi-stationary random excitations according to the various data related to the seismicity and dynamic characteristics of ground at the site, as well as the measure of aseismic safety and the dynamic properties of the structure. This paper, however, is not concerned with the method of constructing a model of earthquake excitations but deals with a method of statistical analysis of the response spectra of quasi-stationary random excitations which are prescribed rather a priori.

In order to discuss strictly the statistical properties of the response spectrum, which is a spectral representation of a non-stationary input excitation in terms of the maximum values of the output responses of a single-degree-of-freedom oscillator with continuously varying frequency parameter suddenly subjected to the input excitation, the probability distribution of the maximum value of the output response in a finite time domain should be found. However, it is very difficult to obtain the analytical expression of the probability distribution of the maximum output response of a dynamic system subjected to a non-stationary random process, even in the case of a single-degree-of-freedom oscillator subjected to simple random excitations.[5),15),16)]

Since the purpose of this paper is to find the statistical properties of the response spectra of a general class of quasi-stationary random excitations, which are applicable to the supposition of a model of earthquake excitations corresponding to the specific seismicity and dynamic characteristics of the ground at the site of a structure, the methods of analysis can not be based only on purely analytical means; they should mainly be based partially analytical and partially experimental techniques.

In this paper it is assumed that the envelope of the quasi-stationary random excitations is given by the product of an arbitrary, deterministic, continuous time-function and a cutoff operator containing the duration time of excitations as a parameter and that the spectral density of the elemental stationary random process is expressed by the product of an arbitrary rational function and a band-limiting operator. The expressions of the mean value and the probable upper and lower limits of the response spectra of the quasi-stationary random excitations are considered by supposing the Gaussian character and the boundedness of the amplitude probability distribution of the random excitations.

2. Response spectra of quasi-stationary random excitations

The fundamental equation of a single-degree-of-freedom linear oscillator subjected to an arbitrary acceleration excitation is given by

$$\left(\frac{d^2}{d\tau^2}+2h\omega\frac{d}{d\tau}+\omega^2\right)\eta(\tau)=f(\tau) \tag{1}$$

in which τ, $f(\tau)$ and $\eta(\tau)$ are time, the acceleration excitation and the relative diplacement of the oscillator, respectively, and ω and h are the natural angular frequency and the critical damping ratio, respectively.

By introducing new frequency and damping parameters defined as

$$h'=\frac{h}{\sqrt{1-h^2}}, \qquad \omega'=\omega\sqrt{1-h^2} \tag{2}$$

where

$$0\leqq h<1, \qquad -\infty<\omega<\infty$$

and the cutoff operator, associated with a finite time domain $R^1{}_{0\tau}$, given by

$$\begin{aligned} &D(\mu\,;\,R^1{}_{0\tau})=s(\mu)-s(\mu-\tau)\\ &R^1{}_{0\tau}=[0,\,\tau] \end{aligned} \tag{3}$$

where $s(\mu)$ is the step-function, the following complex-valued function of τ, ω' and h' is defined:

$$\begin{aligned} A(\tau\,;\,\omega',\,h')&=\int_{-\infty}^{\infty}D(\mu\,;\,R^1{}_{0\tau})f(\mu)\exp(-h'|\omega'|(\tau-\mu))\exp(-j\omega'\mu)d\mu\\ &=\int_0^{\tau}f(\mu)\exp(-h'|\omega'|(\tau-\mu))\exp(-j\omega'\mu)d\mu \end{aligned} \tag{4}$$

By making use of this function the relative displacement $\eta(\tau)$, the relative velocity $\frac{d}{d\tau}\eta(\tau)$ and the absolute acceleration $\frac{d^2}{d\tau^2}\eta(\tau)-f(\tau)$ of the oscillator subjected to the acceleration excitation $f(\tau)$ under the zero initial conditions can be expressed in the following forms:

$$\begin{aligned} &\eta(\tau)=\frac{1}{\omega'}|A|\sin(\omega'\tau+\varphi_D)\\ &\qquad\varphi_D=\arg A=\tan^{-1}\frac{\boldsymbol{I}(A)}{\boldsymbol{R}(A)} \end{aligned} \tag{5}$$

$$\begin{aligned} &\frac{d}{d\tau}\eta(\tau)=\sqrt{1+h'^2}|A|\sin(\omega'\tau+\varphi_V)\\ &\qquad\varphi_V=\varphi_D+\arg(j-h')=\varphi_D+\tan^{-1}\left(-\frac{1}{h'}\right)=\tan^{-1}\frac{\boldsymbol{R}(A)-h'\boldsymbol{I}(A)}{\boldsymbol{I}(A)+h'\boldsymbol{R}(A)} \end{aligned} \tag{6}$$

$$\begin{aligned} &\frac{d^2}{d\tau^2}\eta(\tau)-f(\tau)=\omega'(1+h'^2)|A|\sin(\omega'\tau+\varphi_A)\\ &\qquad\varphi_A=\varphi_D+2\tan^{-1}\left(-\frac{1}{h'}\right)=\tan^{-1}\frac{(1-h'^2)\boldsymbol{I}(A)+2h'\boldsymbol{R}(A)}{(1-h'^2)\boldsymbol{R}(A)-2h'\boldsymbol{I}(A)} \end{aligned} \tag{7}$$

where the symbols $\boldsymbol{R}(A)$ and $\boldsymbol{I}(A)$ denote the real and imaginary parts of the complex-number A, respectively.

From eqs. (5)~(7) the upper bounds of the maximum values of the relative displacement, the relative velocity and the absolute acceleration in the infinite time domain are obtained respectively as:

$$RD(\omega, h)=\sup_{T}|\eta(\tau)|\leqq\frac{1}{\omega'}\sup_{T}|A(\tau\,;\,\omega',\,h')| \tag{8}$$

$$RV(\omega, h)=\sup_{T}\ \frac{d}{d\tau}\eta(\tau)\ \leqq\sqrt{1+h'^2}\sup_{T}|A(\tau\,;\,\omega',\,h')| \tag{9}$$

$$AA(\omega, h)=\sup_{T}\ \frac{d^2}{d\tau^2}\eta(\tau)-f(\tau)\ \leqq\omega'(1+h'^2)\sup_{T}|A(\tau\,;\,\omega',\,h')| \tag{10}$$

where

$$T=R^1{}_{0\infty}=[0,\ \infty)$$

If the absolute value of the complex-valued function $A(\tau\,;\,\omega',\,h')$ is a slowly varying time-function compared with the sinusoidal function $\sin\omega'\tau$, each right-hand side of eqs. (8)~(10) may be approximately equal to the least upper bound, that is, the upper limit of the relevant response of the oscillator.

The velocity response spectrum of an acceleration excitation which is a kind of spectral representation of the non-stationary input excitation is defined by the following equation:[17),18)]

$$S_V(\omega, h)=\sup_{T}|J(\tau\,;\,\omega,\,h)| \tag{11}$$

where ω and h are the original frequency and damping parameters, respectively and

$$\begin{aligned}J(\tau\,;\,\omega,\,h)&=\boldsymbol{I}(A(\tau\,;\,\omega,\,h)\exp(j\omega\tau))\\&=|A(\tau\,;\,\omega,\,h)|\sin(\omega\tau+\arg A(\tau\,;\,\omega,\,h))\\&=\int_0^{\tau}f(\mu)\exp(-h|\omega|(\tau-\mu))\sin\omega(\tau-\mu)d\mu\end{aligned} \tag{12}$$

which is the output response of the linear system having the impulsive response $g(\tau)=\exp(-h|\omega|\tau)\sin\omega\tau$, subjected to an input excitation $f(\tau)$ under the zero initial condition.

From eqs. (11) and (12) the upper bound of the velocity response spectrum is obtained as follows:

$$\begin{aligned}&S_V(\omega, h)\leqq\sup_{T}|A(\tau\,;\,\omega,\,h)|\\&A(\tau\,;\,\omega,\,h)=\int_0^{\tau}f(\mu)\exp(-h|\omega|(\tau-\mu))\exp(-j\omega\mu)d\mu\end{aligned} \tag{13}$$

Similarly as in the previous case, if the function $|A(\tau\,;\,\omega,\,h)|$ is a slowly varying time-function compared with $\sin\omega\tau$, the function $|A(\tau\,;\,\omega,\,h)|$ gives the envelope of the function $J(\tau\,;\,\omega,\,h)$ defined by eq. (12) and the right-hand side of the first equation of (13) may be approximately equal to the upper limit of the response spectrum defined by eq. (11).

The maximum values of the relative displacement, the relative velocity and the absolute acceleration of the linear oscillator are approximately expressed respectively as follows, by making use of the above defined velocity spectrum and the frequency and damping parameters defined by eq. (2):

$$RD(\omega, h, \tau_d) \doteqdot \frac{1}{\omega'} S_V(\omega', h', \tau_d) \tag{14}$$

$$RV(\omega, h, \tau_d) \doteqdot \sqrt{1+h'^2} S_V(\omega', h', \tau_d) \tag{15}$$

$$AA(\omega, h, \tau_d) \doteqdot \omega'(1+h'^2) S_V(\omega', h', \tau_d) \tag{16}$$

In a case where the damping parameter h is sufficiently small compared with unity the following approximations are valid :

$$h' \doteqdot h, \qquad \omega' \doteqdot \omega, \qquad \sqrt{1+h'^2} \doteqdot 1 \tag{17}$$

hence eqs. (14)$\sim$(16) are reduced respectively to the following forms :

$$RD(\omega, h) \doteqdot \frac{1}{\omega} S_V(\omega, h) \tag{18}$$

$$RV(\omega, h) \doteqdot S_V(\omega, h) \tag{19}$$

$$AA(\omega, h) \doteqdot \omega S_V(\omega, h) \tag{20}$$

The right-hand sides of eqs. (18) and (20) are called the displacement response spectrum and the acceleration response spectrum of the acceleration excitation $f(\tau)$, respectively.[17),18)]

Now, a quasi-stationary random process is defined as the product of a deterministic time-function and a stationary random process as follows :

$$f(\tau) = a(\tau)\psi(\tau) \tag{21}$$

where $\psi(\tau)$ is a sample function of a stationary random process with zero mean and $a(\tau)$ is an arbitrary deterministic time-function which gives the envelope of a quasi-stationary random process if $a(\tau)$ is a slowly varying time-function compared with $\psi(\tau)$.

To estimate definitely the effect of the duration time τ_d of the random excitations on the response spectra, the random process expressed by the product of the quasi-stationary random process defined by eq. (21) and the cutoff operator defined by eq. (3) are considered hereafter, namely

$$D(\tau\,;\,R^1{}_{0\tau_d})f(\tau) = D(\tau\,;\,R^1{}_{0\tau_d})a(\tau)\psi(\tau) \tag{22}$$

Of course, the time-function defined by the above equation is also a sample function of the quasi-stationary random process having the deterministic function $D(\tau\,;\,R^1{}_{0\tau_d})a(\tau)$. From eqs. (10) and (11), the velocity response spectrum of the quasi-stationary random excitation defined by eq. (22) is expressed as follows :

$$S_V(\omega, h, \tau_d) = \sup_{\tau} |J_\xi(\tau\,;\,\omega, h, \tau_d)| \tag{23}$$

$$J_\xi(\tau\,;\,\omega, h, \tau_d) = |A_\xi(\tau\,;\,\omega, h, \tau_d)| \sin(\omega\tau + \arg A_\xi(\tau\,;\,\omega, h, \tau_d)) \tag{24}$$

$$A_\xi(\tau\,;\,\omega, h, \tau_d) = \int_0^{\tau_m} f(\mu)\exp(-h|\omega|(\tau-\mu))\exp(-j\omega\mu)\,d\mu \tag{25}$$

$$\tau_m = \min(\tau, \tau_d) \tag{26}$$

3. Energy and power spectral densities of the modified quasi-stationary random process

In connection with eqs. (24) and (25) the modified quasi-stationary random process $\xi(\mu, \tau; \omega, h, \tau_d)$ associated with the quasi-stationary random process $f(\tau) = a(\tau)\phi(\tau)$ is defined by the following equation:

$$\begin{aligned}\xi(\mu, \tau; \omega, h, \tau_d) &= D(\mu; R'_{0\tau})D(\mu; R'_{0\tau_d})\exp(-h|\omega|(\tau-\mu))f(\mu)\\ &= D(\mu; R'_{0\tau_m})\exp(-h|\omega|(\tau-\mu))f(\mu)\end{aligned} \tag{27}$$

The energy spectral density $S_{E\xi}(\tau; \omega, h, \tau_d)$ of the modified quasi-stationary random process defined by eq. (27) is given by the ensemble average of the squared absolute value of the Fourier transform of $\xi(\mu, \tau; \omega, h, \tau_d)$ as follows:[3]

$$S_{E\xi}(\tau; \omega, h, \tau_d) = E|A_\xi(\tau; \omega, h, \tau_d)|^2 \tag{28}$$

where

$$\begin{aligned}A_\xi(\tau; \omega, h, \tau_d) &= \int_{-\infty}^{\infty}\xi(\mu, \tau; \omega, h, \tau_d)\exp(-j\omega\mu)d\mu\\ &= \int_0^{\tau_m} f(\mu)\exp(-h|\omega|(\tau-\mu))\exp(-j\omega\mu)d\mu\end{aligned}$$

In eq. (28) the symbol E denotes the ensemble average.

The power spectral density $S_{H\xi}(\tau; \omega, h, \tau_d)$ of the modified quasi-stationary random process is defined by the following equation in the similar form introduced by D. G. Lampard:[3,19]

$$S_{H\xi}(\tau; \omega, h, \tau_d) = S_{E\xi\tau}{}^{(1)}(\tau; \omega, h, \tau_d) \tag{29}$$

where ${}_\tau{}^{(i)}$ denotes the ith order partial differentiation with respect to τ.

The energy and power spectral densities which are defined as the real-valued functions of ω and τ by eqs. (28) and (29), respectively, are available to the general class of non-stationary random process, and the former is a real positive-valued function but the latter is not always positive for the non-stationary random processes containing the quasi-stationary random process.[3] Integrating eq. (29) with the zero initial condition the energy spectral density is expressed as follows;

$$S_{E\xi}(\tau; \omega, h, \tau_d) = \int_0^\tau S_{H\xi}(\tau; \omega, h, \tau_d)d\tau \tag{30}$$

It should be noticed that the frequency parameter ω contained in the modified quasi-stationary random process $\xi(\mu, \tau; \omega, h, \tau_d)$ is essentially independent of the frequency parameter of the Fourier transform. Hence if these parameters are distinguished from each other, the inverse Fourier transform of eq. (28) gives an integral of the co-variance of the modified quasi-stationary random process. In particular, the value of this integral at zero, which means the integral of the variance over the time domain $R'_{0\tau_m}$ or the integral of the energy spectral density with respect to the frequency parameter of the Fourier transform devided by 2π, gives the mean value of the total energy of the modified quasi-stationary random process.[3]

In general, supposing that random time-functions $f(\tau)$ belong to a non-stationary random process the co-variance of the modified non-stationary random process as given by eq. (27) is expressed as follows:

$$K_\xi(\mu_1, \mu_2)$$
$$=D(\mu_1\,;\, R^1{}_{0\tau_m})D(\mu_2\,;\, R^1{}_{0\tau_m})\exp(-2h|\omega|\tau)\exp(h|\omega|(\mu_1+\mu_2))K_f(\mu_1, \mu_2) \quad (31)$$

Transforming the variables μ_1 and μ_2 into ν and κ by the equations,

$$\nu=\mu_1-\mu_2, \qquad \kappa=\mu_2$$

the energy spectral density of the modified non-stationary random process is expressed as follows:[3]

$$S_{E\xi}(\tau\,;\,\omega, h, \tau_d)=\exp(-2h|\omega|\tau)\int_0^{\tau_m}d\kappa\exp(2h|\omega|\kappa)\int_{-\kappa}^{\tau_m-\kappa}d\nu K_f(\nu+\kappa, \kappa)\cdot\exp(h|\omega|\nu)\exp(-j\omega\nu) \quad (32)$$

where

$$K_f(\nu+\kappa, \kappa)=E(f(\nu+\kappa)f(\kappa)) \quad (33)$$

is the co-variance of the non-stationary random process $f(\tau)$.

As the co-variance of a non-stationary random process is expressed as the inverse Fourier transform of the one-dimensional total spectral density $S_f(\omega, \kappa)$ of the process, namely

$$K_f(\nu+\kappa, \kappa)=\frac{1}{2\pi}\int_{-\infty}^{\infty}S_f(\omega', \kappa)\exp(j\omega'\nu)d\omega' \quad (34)$$

eq. (32) can be rewritten as follows:[3]

$$\begin{aligned}S_{E\xi}(\tau\,;\,\omega, h, \tau_d)&=\frac{\exp(-2h|\omega|\tau)}{2\pi}\int_0^{\tau_m}d\kappa\exp(2h|\omega|\kappa)\int_{-\infty}^{\infty}d\omega' S_f(\omega', \kappa)\\&\quad\cdot\int_{-\kappa}^{\tau_m-\kappa}d\nu\exp\{(h|\omega|-j(\omega-\omega'))\nu\}\\&=\frac{\exp(-2h|\omega|\tau)}{2\pi}\int_{-\infty}^{\infty}d\omega'\frac{\exp\{(h|\omega|-j(\omega-\omega'))\tau_m\}-1}{h|\omega|-j(\omega-\omega')}\\&\quad\cdot\int_0^{\tau_m}d\kappa S_f(\omega', \kappa)\exp\{(h|\omega|+j(\omega-\omega'))\kappa\}\end{aligned} \quad (35)$$

Here, by considering that the random functions $f(\tau)$ belong to a quasi-stationary random process, the co-variance and the one-dimensional total spectral density of the modified quasi-stationary process are expressed in the following forms respectively:

$$K_f(\nu+\kappa, \kappa)=a(\nu+\kappa)a(\kappa)R_\phi(\nu) \quad (36)$$

$$\begin{aligned}S_f(\omega, \kappa)&=\int_{-\infty}^{\infty}K_f(\nu+\kappa, \kappa)\exp(-j\omega\nu)d\nu\\&=\frac{1}{2\pi}a(\kappa)\exp(j\omega\kappa)\int_{-\infty}^{\infty}d\mu\exp(-j\mu\kappa)A(j(\omega-\mu))S_\phi(\mu)\\&=\frac{1}{2\pi}a(\kappa)\int_{-\infty}^{\infty}\exp(j\mu\kappa)A(j\mu)S_\phi(\omega-\mu)d\mu\end{aligned} \quad (37)$$

in which $R_\phi(\tau)$ and $S_\phi(\omega)$ are the auto-correlation function and the power spectral density of the stationary random process $\phi(\tau)$ respectively and $A(j\omega)$ is the Fourier transform of the deterministic function $a(\tau)$. Among these quantities the following relations are valid:

$$S_\psi(\omega)=\int_{-\infty}^{\infty} R_\psi(\lambda)\exp(-j\omega\lambda)d\lambda, \qquad R_\psi(\lambda)=\frac{1}{2\pi}\int_{-\infty}^{\infty} S_\psi(\mu)\exp(j\mu\lambda)d\mu \tag{38}$$

$$A(j\omega)=\int_{-\infty}^{\infty} a(\kappa)\exp(-j\omega\kappa)d\kappa \tag{39}$$

By making use of eq. (37) the second integral contained in eq. (35) can be expressed as

$$\begin{aligned}&\int_0^{\tau_m} S_f(\omega', \kappa)\exp\{(h|\omega|+j(\omega-\omega'))\kappa\}d\kappa\\ &\quad=\frac{1}{2\pi}\int_{-\infty}^{\infty} d\mu S_\psi(\mu)A(j(\omega'-\mu))\int_0^{\tau_m} d\kappa a(\kappa)\exp\{(h|\omega|+j(\omega-\mu))\kappa\}\end{aligned} \tag{40}$$

Moreover the second integral of the above equation is written in the following form by using of eq. (39) :

$$\begin{aligned}&\int_0^{\tau_m} d\kappa a(\kappa)\exp\{(h|\omega|+j(\omega-\mu))\kappa\}\\ &\quad=\frac{1}{2\pi}\int_{-\infty}^{\infty} d\nu A(j(\nu-\omega))\frac{\exp\{(h|\omega|+j(\nu-\mu))\tau_m\}-1}{h|\omega|+j(\nu-\mu)}\end{aligned} \tag{41}$$

Hence the energy spectral density of the modified quasi-stationary random process defined by eq. (27) is expressed as follows:

$$S_{E\xi}(\tau\,;\,\omega,\,h,\,\tau_d)=\frac{1}{2\pi}\int_{-\infty}^{\infty} S_\psi(\mu)|c(\mu,\,\tau\,;\,\omega,\,h,\,\tau_d)|^2 d\mu \tag{42}$$

where

$$c(\mu,\,\tau\,;\,\omega,\,h,\,\tau_d)=\exp(-h|\omega|\tau)B(\mu\,;\,\omega,\,h,\,\tau_m) \tag{43}$$

$$\begin{aligned}B(\mu\,;\,\omega,\,h,\,\tau_m)&=\int_0^{\tau_m} a(\kappa)\exp\{(h|\omega|-j(\omega-\mu))\kappa\}d\kappa\\ &=\frac{1}{2\pi}\int_{-\infty}^{\infty} A(j(\omega'-\mu))\frac{\exp\{(h|\omega|+j(\omega'-\omega))\tau_m\}-1}{h|\omega|+j(\omega'-\omega)}d\omega'\\ &=\frac{1}{2\pi}\int_{-\infty}^{\infty} A(j\omega')\frac{\exp\{(h|\omega|-j(\omega-\mu-\omega'))\tau_m\}-1}{h|\omega|-j(\omega-\mu-\omega')}d\omega'\end{aligned} \tag{44}$$

in which it is noted that the following relation is valid :

$$B^*(\mu\,;\,\omega,\,h,\,\tau_m)=B(\omega,\,\mu,\,h,\,\tau_m) \tag{45}$$

where superscript $*$ denotes the complex-conjugate.

By making use of eqs. (29) and (42) the power spectral density of the modified quasi-stationary random process is expressed by

$$\begin{aligned}S_{H\xi}(\tau\,;\,\omega,\,h,\,\tau_d)&=\frac{1}{2\pi}\int_{-\infty}^{\infty} d\mu S_\psi(\mu)\,\frac{\partial}{\partial\tau}\,|c(\mu,\,\tau\,;\,\omega,\,h,\,\tau_d)|^2\\ &=\frac{1}{\pi}\int_{-\infty}^{\infty} d\mu S_\psi(\mu)\boldsymbol{R}(c_\tau^{(1)}(\mu,\,\tau\,;\,\omega,\,h,\,\tau_d)c^*(\mu,\,\tau\,;\,\omega,\,h,\,\tau_d))\end{aligned} \tag{46}$$

Since from eqs. (26), (43) and (44) the following equations are obtained;

$$\begin{aligned}&c_\tau^{(1)}(\mu,\,\tau\,;\,\omega,\,h,\,\tau_d)c^*(\mu,\,\tau\,;\,\omega,\,h,\,\tau_d)\\ &\quad=\exp(-2h|\omega|\tau)\Big(a(\tau)\exp\{(h|\omega|-j(\omega-\mu))\tau\}\int_0^{\tau} a(\kappa)\exp\{(h|\omega|+j(\omega-\mu))\kappa\}d\kappa\\ &\qquad-h|\omega|\Big|\int_0^{\tau} a(\kappa)\exp\{(h|\omega|-j(\omega-\mu))\kappa\}d\kappa\Big|^2\Big), \qquad 0\leqq\tau<\tau_d\end{aligned} \tag{47}$$

$$
\begin{aligned}
&c_\tau^{(1)}(\mu,\ \tau\ ;\ \omega,\ h,\ \tau_d)c^*(\mu,\ \tau\ ;\ \omega,\ h,\ \tau_d)\\
&\quad=-h|\omega|\exp(-2h|\omega|\tau)\int_0^{\tau_d}a(\kappa)\exp\{(h|\omega|-j(\omega-\mu))\kappa\}^2\leqq 0, \qquad \tau>\tau_d
\end{aligned}
\tag{48}
$$

and

$$
\begin{aligned}
&R(c_\tau^{(1)}(\mu,\ \tau\ ;\ \omega,\ h,\ \tau_d)c^*(\mu,\ \tau\ ;\ \omega,\ h,\ \tau_d))\\
&\quad=a(\tau)\int_0^{\tau}a(\kappa)\exp(-h|\omega|(\tau-\kappa))\cos(\omega-\mu)(\tau-\kappa)d\kappa\\
&\quad-h|\omega|\int_0^{\tau}a(\kappa)\exp\{-h|\omega|(\tau-\kappa)-j(\omega-\mu)\kappa\}d\kappa^2, \quad 0\leqq\tau<\tau_d
\end{aligned}
\tag{49}
$$

the power spectral density of the modified quasi-stationary random process defined by eq. (27) is expressed as follows:

$$
\begin{aligned}
S_{H\xi}(\tau\ ;\ \omega,\ h,\ \tau_d)=&\frac{1}{\pi}\int_{-\infty}^{\infty}S_\psi(\mu)(a(\tau)Rb(\mu,\ \tau\ ;\ \omega,\ h)\\
&-h|\omega||b(\mu,\ \tau\ ;\ \omega,\ h)|^2)d\mu, \qquad 0\leqq\tau<\tau_d
\end{aligned}
\tag{50}
$$

$$
\begin{aligned}
S_{H\xi}(\tau\ ;\ \omega,\ h,\ \tau_d)=&-\frac{h|\omega|\exp(-2h|\omega|(\tau-\tau_d))}{\pi}\int_{-\infty}^{\infty}S_\psi(\mu)|b(\mu,\ \tau_d\ ;\ \omega,\ h)|^2d\mu\leqq 0\\
&\text{for}\quad h\geqq 0, \qquad \tau>\tau_d
\end{aligned}
\tag{51}
$$

where

$$
\begin{aligned}
b(\mu,\ \tau\ ;\ \omega,\ h)&=\int_0^{\tau}a(\kappa)\exp\{-(h|\omega|+j(\omega-\mu))(\tau-\kappa)\}d\kappa\\
&=\int_0^{\tau}a(\tau-\kappa)\exp\{-(h|\omega|+j(\omega-\mu))\kappa\}d\kappa\\
&=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\nu A(j\nu)\exp(j\nu\tau)\frac{1-\exp\{-(h|\omega|+j(\omega+\nu-\mu))\tau\}}{h|\omega|+j(\omega+\nu-\mu)}
\end{aligned}
\tag{52}
$$

In deriving eqs. (47) and (48) the following identity is considered:

$$
\begin{aligned}
&\int_0^{\tau}a(\kappa)\exp\{-h|\omega|(\tau-\kappa)-j(\omega-\mu)\kappa\}d\kappa^2\\
&\quad=\int_0^{\tau}a(\kappa)\exp\{-h|\omega|(\tau-\kappa)-j(\omega-\mu)(\tau-\kappa)\}d\kappa^2
\end{aligned}
$$

By making use of the equations which are

$$
\begin{aligned}
&Rb(\mu,\ \tau\ ;\ \omega,\ h)=\int_0^{\tau}a(\tau-\kappa)\exp(-h|\omega|\kappa)\cos(\omega-\mu)\kappa d\kappa\\
&|b(\mu,\ \tau\ ;\ \omega,\ h)|^2=\int_0^{\tau}d\kappa_1\int_0^{\tau}d\kappa_2 a(\tau-\kappa_1)a(\tau-\kappa_2)\\
&\qquad\cdot\exp(-h|\omega|(\kappa_1+\kappa_2))\exp(-j(\omega-\mu)(\kappa_1-\kappa_2))
\end{aligned}
\tag{53}
$$

and

$$
\begin{aligned}
&\frac{1}{2\pi}\int_{-\infty}^{\infty}S_\psi(\mu)\cos\mu\kappa=R_\psi(\kappa), \qquad \frac{1}{2\pi}\int_{-\infty}^{\infty}S_\psi(\mu)\sin\mu\kappa=0\\
&\frac{1}{2\pi}\int_{-\infty}^{\infty}S_\psi(\mu)\exp(j\mu(\kappa_1-\kappa_2))d\mu=R_\psi(\kappa_1-\kappa_2)
\end{aligned}
\tag{54}
$$

the power spectral density of the modified quasi-stationary random process is also expressed in the following forms:

$$
S_{H\xi}(\tau\ ;\ \omega,\ h,\ \tau_d)=2\Big(a(\tau)\int_0^{\tau}a(\tau-\kappa)\exp(-h|\omega|\kappa)R_\psi(\kappa)\cos\omega\kappa d\kappa
$$

$$-h|\omega|\int_0^\tau d\kappa_1\int_0^\tau d\kappa_2 a(\tau-\kappa_1)a(\tau-\kappa_2)\exp(-h|\omega|(\kappa_1+\kappa_2))$$
$$\cdot R_\phi(\kappa_1-\kappa_2)\exp(-j\omega(\kappa_1-\kappa_2))\Big), \qquad 0\leqq\tau<\tau_d \tag{55}$$

$$S_{H\xi}(\tau\,;\,\omega,\,h,\,\tau_d)=-2h|\omega|\exp(-2h|\omega|(\tau-\tau_d))\int_0^{\tau_d}d\kappa_1\int_0^{\tau_d}d\kappa_2 a(\tau_d-\kappa_1)a(\tau_d-\kappa_2)$$
$$\cdot\exp(-h|\omega|(\kappa_1+\kappa_2))R_\phi(\kappa_1-\kappa_2)\exp(-j\omega(\kappa_1-\kappa_2)), \qquad \tau>\tau_d \tag{56}$$

Between the quantities defined by eqs. (43) and (52), the following identity is valid in the time domain $[0,\,\tau_d]$:

$$|c(\mu,\,\tau\,;\,\omega,\,h,\,\tau_d)|^2=|b(\mu,\,\tau\,;\,\omega,\,h)|^2, \qquad 0\leqq\tau\leqq\tau_d \tag{57}$$

Hence for the time domain $[0,\,\tau_d)$ the energy and power spectral densities of the modified quasi-stationary random process are expressed as follows:

$$S_{E\xi}(\tau\,;\,\omega,\,h,\,\tau_d)=\frac{1}{2\pi}\int_{-\infty}^{\infty}S_\phi(\mu)|b(\mu,\,\tau\,;\,\omega,\,h)|^2d\mu, \qquad 0\leqq\tau\leqq\tau_d \tag{58}$$

$$S_{H\xi}(\tau\,;\,\omega,\,h,\,\tau_d)=\frac{a(\tau)}{\pi}\int_{-\infty}^{\infty}S_\phi(\mu)Rb(\mu,\,\tau\,;\,\omega,\,h)d\mu$$
$$-2h|\omega|S_{E\xi}(\tau\,;\,\omega,\,h,\,\tau_d)d\mu, \qquad 0\leqq\tau<\tau_d \tag{59}$$

From eqs. (29) and (59) the following differential and integral equations are obtained:

$$\left(\frac{\partial}{\partial\tau}+2h|\omega|\right)S_{E\xi}(\tau\,;\,\omega,\,h,\,\tau_d)=f(\tau\,;\,\omega,\,h), \qquad 0\leqq\tau<\tau_d \tag{60}$$

$$\left(1+2h|\omega|\int_0^\tau d\tau\right)S_{H\xi}(\tau\,;\,\omega,\,h,\,\tau_d)=f(\tau\,;\,\omega,\,h), \qquad 0\leqq\tau<\tau_d \tag{61}$$

where

$$f(\tau\,;\,\omega,\,h)=\frac{a(\tau)}{\pi}\int_{-\infty}^{\infty}S_\phi(\mu)Rb(\mu,\,\tau\,;\,\omega,\,h)d\mu \tag{62}$$

Solving eq. (60) under the zero initial condition the energy spectral density of the modified quasi-stationary random process for the time domain $[0,\,\tau_d]$ can be expressed by the following convolution integral associated with the time domain $[0,\,\tau]$.

$$\begin{aligned}S_{E\xi}(\tau\,;\,\omega,\,h,\,\tau_d)&=\exp(-2h|\omega|\tau)*f(\tau\,;\,\omega,\,h)\\&=\int_0^\tau\exp(-2h|\omega|(\tau-\kappa))f(\kappa\,;\,\omega,\,h)d\kappa, \qquad 0\leqq\tau\leqq\tau_d\end{aligned} \tag{63}$$

And also, by solving eq. (61) or from eqs. (29) and (63) the power spectral density of the modified quasi-stationary random process for the time domain $[0,\,\tau_d)$ is written in the following form:

$$\begin{aligned}S_{H\xi}(\tau\,;\,\omega,\,h,\,\tau_d)&=(\delta(\tau)-2h|\omega|\exp(-2h|\omega|\tau))*f(\tau\,;\,\omega,\,h)\\&=f(\tau\,;\,\omega,\,h)-2h|\omega|\exp(-2h|\omega|\tau)*f(\tau\,;\,\omega,\,h)\\&=f(\tau\,;\,\omega,\,h)-2h|\omega|\int_0^\tau\exp(-2h|\omega|(\tau-\kappa))f(\kappa\,;\,\omega,\,h)d\kappa\end{aligned} \tag{64}$$

where $\delta(\tau)$ is Dirac's delta-function.

On the other hand, by substituting the first equation of (53) in eq. (62) the non-homogeneous term of eqs. (60) and (61) is given by the following equation:

$$
\begin{aligned}
f(\tau\,;\,\omega,\,h) &= \frac{a(\tau)}{\pi}\int_{-\infty}^{\infty} d\mu S_\phi(\mu)\int_0^\tau a(\kappa)\exp(-h|\omega|(\tau-\kappa))\cos(\omega-\mu)(\tau-\kappa)d\kappa \\
&= \frac{a(\tau)}{\pi}\int_0^\tau d\kappa a(\tau-\kappa)\exp(-h|\omega|\kappa)\int_{-\infty}^{\infty} d\mu S_\phi(\mu)\cos(\omega-\mu)\kappa \\
&= 2a(\tau)[a(\tau)*\exp(-h|\omega|\tau)\cos\omega\tau R_\phi(\tau)]
\end{aligned} \tag{65}
$$

In the above, the energy and power spectral densities of the modified quasi-stationary random process defined by eq. (27) are obtained as the functions of time τ and parameters ω, h and τ_d containing the deterministic function $a(\tau)$ and the power spectral density $S_\phi(\omega)$ or the auto-correlation function $R_\phi(\tau)$ of the stationary process $\phi(\tau)$ by which the original quasi-stationary random process is defined as in eq. (21).

As shown in eqs. (13) and (23), the maximum absolute value of the Fourier transform $\sup_\tau |A_\xi(\tau\,;\,\omega,\,h,\,\tau_d)|$ of the modified quasi-stationary random excitation is approximately equal to the response spectrum of the acceleration excitation $D(\tau\,;\,R^1{}_{0\tau_d})a(\tau)\phi(\tau)$ if $|A_\xi(\tau\,;\,\omega,\,h,\,\tau_d)|$ is a slowly varying time-function compared with $\sin\omega\tau$. In this sense the energy spectral density $S_{E\xi}(\tau\,;\,\omega,\,h,\,\tau_d)$, defined as the ensemble average of the squared absolute value $|A_\xi(\tau\,;\,\omega,\,h,\,\tau_d)|^2$, and the power spectral density $S_{H\xi}(\tau\,;\,\omega,\,h,\,\tau_d)$, defined as the time derivative of $S_{E\xi}(\tau\,;\,\omega,\,h,\,\tau_d)$, may be related to the average value of the response spectra $ES_V(\omega,\,h,\,\tau_d)$ of the quasi-stationary random excitations.

4. Expressions of the mean value and the upper and lower limits of the response spectra of the quasi-stationary random excitations

From eqs. (23) and (28) the mean value of the response spectra of the quasi-stationary random excitations defined by eq. (22) is approximately expressed in the following form when the absolute value of the Fourier transform $|A_\xi(\tau\,;\,\omega,\,h,\,\tau_d)|$ of the modified quasi-stationary random process is a slowly varying time-function compared with the sinusoidal function $\sin\omega\tau$:

$$
ES_V(\omega,\,h,\,\tau_d) = E\sup_\tau |J_\xi(\tau\,;\,\omega,\,h,\,\tau_d)| \doteqdot E\sup_\tau |A_\xi(\tau\,;\,\omega,\,h,\,\tau_d)| \tag{66}
$$

Since the time-function $J_\xi(\tau\,;\,\omega,\,h,\,\tau_d)$ defined by eq. (24) is the output response of a linear system, the probability density distribution of the peak amplitude of $J_\xi(\tau\,;\,\omega,\,h,\,\tau_d)$ may be given by the following formula if the stationary random process $\phi(\tau)$ which defines the quasi-stationary random process as in eq. (22) is Gaussian:[20]

$$
\begin{aligned}
p(|A|\,;\,\tau) = \exp\Big(-\frac{|A|^2}{2K_{JJ}}\Big)\Big[&\frac{|A|}{K_{JJ}}\exp\Big(-\frac{\rho^2|A|^2}{2(1-\rho^2)K_{JJ}}\Big) \\
&+\frac{\rho}{\sqrt{1-\rho^2}}\sqrt{\frac{\pi}{2K_{JJ}}}\Big(\frac{|A|^2}{K_{JJ}}-1\Big)\mathrm{erf}\Big(\frac{\rho|A|}{\sqrt{2(1-\rho^2)K_{JJ}}}\Big)\Big]
\end{aligned} \tag{67}
$$

in which

$$
\rho = \frac{K_{J\dot{J}}}{\sqrt{K_{JJ}K_{\dot{J}\dot{J}}}}, \qquad \mathrm{erf}\,x = \frac{2}{\sqrt{\pi}}\int_0^x e^{-\nu^2}d\nu \tag{68}
$$

and $|A|$ denotes the peak amplitude, K_{JJ} and $K_{\dot{J}\dot{J}}$ are the variances of J and $\dot{J}=\frac{\partial}{\partial\tau}J$, respectively, and $K_{J\dot{J}}$ is the co-variance between J and $\dot{J}$ at time τ.

By introducing the non-dimensional peak amplitude defined by

$$\zeta = \frac{|A|}{\sqrt{K_{JJ}}} \tag{69}$$

the probability density distribution of ζ is given by

$$\begin{aligned}\bar{p}(\zeta\,;\,\tau) &= p(\sqrt{K_{JJ}}\zeta\,;\,\tau)\sqrt{K_{JJ}} \\ &= \exp\left(-\frac{\zeta^2}{2}\right)\left[\zeta \exp\left(-\frac{\xi^2\zeta^2}{2}\right) + \xi\sqrt{\frac{\pi}{2}}(\zeta^2-1)\operatorname{erf}\left(\frac{\xi\zeta}{\sqrt{2}}\right)\right]\end{aligned} \tag{70}$$

where

$$\xi = \frac{\rho}{\sqrt{1-\rho^2}} = \frac{K_{JJ}}{\sqrt{K_{JJ}K_{JJ} - (K_{JJ})^2}} \tag{71}$$

If $|A_\xi|$ is the envelope of a random time-function J_ξ the amplitude probability density distribution of $|A_\xi|$ may be approximately given by the probability density distribution of the peak amplitude $|A|$ of J_ξ. Hence, the mean and the square mean and the variance of $|A_\xi|$ are obtained respectively by using the probability density distribution of the peak amplitude given by eq. (67) or (70) as follows:

$$E|A_\xi| = \int_0^\infty |A_\xi| p(|A_\xi|;\,\tau)d|A_\xi| = \sqrt{K_{JJ}}\int_0^\infty \zeta\bar{p}(\zeta\,;\,\tau)d\zeta = \sqrt{K_{JJ}}E(\zeta) \tag{72}$$

$$E|A_\xi|^2 = \int_0^\infty |A_\xi|^2 p(|A_\xi|\,;\,\tau)d|A_\xi| = K_{JJ}\int_0^\infty \zeta^2\bar{p}(\zeta\,;\,\tau)d\zeta = K_{JJ}E(\zeta^2) \tag{73}$$

$$\begin{aligned}V|A_\xi| &= \int_0^\infty (|A_\xi| - E|A_\xi|)^2 p(|A_\xi|;\,\tau)d|A_\xi| = K_{JJ}\int_0^\infty (\zeta - E\zeta)^2\bar{p}(\zeta\,;\,\tau)d\zeta \\ &= K_{JJ}V(\zeta) = K_{JJ}\{E(\zeta^2) - (E(\zeta))^2\}\end{aligned} \tag{74}$$

in which the mean, the square mean and the variance of the non-dimensional random variable ζ are expressed as the even functions of the dimensionless quantity ξ defined by eq. (71) as follows:

$$E(\zeta) = \sqrt{\frac{\pi}{2}}\sqrt{1+\xi^2} \tag{75}$$

$$E(\zeta^2) = 2(1+\xi\tan^{-1}\xi) \tag{76}$$

$$V(\zeta) = \frac{4-\pi}{2} + \xi\left(2\tan^{-1}\xi - \frac{\pi}{2}\xi\right) \tag{77}$$

It is clearly a contradiction that when $|\xi|$ becomes large eq. (77) takes a negative value. This may originate from the inconsistency that when $|\xi|$ is large and ζ is small, the probability density given by eq. (70) becomes negative. The zero of eq. (77) ξ_0 at which the variance of ζ vanishes and the corresponding normalized co-variance ρ_0 are calculated respectively as follows:

$$\xi_0 = \pm 1.261, \qquad \rho_0 = \pm 0.785 \tag{78}$$

Therefore it is necessary to limit the range of ξ or ρ for which the probability density distribution given by eq. (70) can be adopted, for instance,

$$|\xi| \leqq 1.0 \quad \text{i.e.,} \quad |\rho| \leqq \frac{1}{\sqrt{2}} = 0.717 \tag{79}$$

If the random variable $|A_\xi|$ is bounded the bounded, positive number $\bar{\lambda}$ exists, which satisfies the following equation:

$$\sup_{E}|A_\xi| = E|A_\xi| + \bar{\lambda}\sqrt{V|A_\xi|}, \qquad \bar{\lambda} \geqq 0 \tag{80}$$

The quantity $\bar{\lambda}$ means the maximum value of the normalized random variable associated with $|A_\xi|$. If the absolute value of the minimum $|\underline{\lambda}|$ of the normalized random variable is not greater than $\bar{\lambda}$, namely

$$\bar{\lambda} \geqq |\underline{\lambda}|, \text{i.e.,} \quad (|A_\xi| - E|A_\xi|)_{max} \geqq |(|A_\xi| - E|A_\xi|)_{min}| \tag{81}$$

the following inequality is valid:

$$\bar{\lambda} \geqq 1 \tag{82}$$

With regard to the envelope of the peak amplitude eq. (81) accordingly eq. (82) seems to be valid.

On the other hand, the following upper and lower bounds for the mean value of response spectra exist.

$$\sup_{T}\sup_{E}|A_\xi| = \sup_{E}\sup_{T}|A_\xi| \geqq ES_V \doteqdot E\sup_{T}|A_\xi| \geqq \sup_{T} E|A_\xi| \tag{83}$$

Here, defining the following functions of time τ, which depend on the probability density distribution of the envelope $|A_\xi|$, by the equations,

$$d_1(p(|A_\xi|\,;\,\tau)) = \frac{E|A_\xi|}{\sqrt{E|A_\xi|^2}} = \frac{E(\zeta)}{\sqrt{E(\zeta^2)}}, \qquad d_2(p(|A_\xi|\,;\,\tau)) = \sqrt{\frac{V|A_\xi|}{E|A_\xi|^2}} = \sqrt{\frac{V(\zeta)}{E(\zeta^2)}} \tag{84}$$

and rewriting $\bar{\lambda} = \bar{\lambda}(p(|A_\xi|\,;\,\tau))$, eq. (83) is reduced to the following form by making use of eq. (80):

$$\begin{aligned} &\sup_{T}[\{d_1(p(|A_\xi|\,;\,\tau)) + \bar{\lambda}(p(|A_\xi|\,;\,\tau))d_2(p(|A_\xi|\,;\,\tau))\}\sqrt{E|A_\xi|^2}] \\ &\quad \geqq ES_V(\omega,\,h,\,\tau_d) \doteqdot E\sup_{T}|A_\xi| \geqq \sup_{T} d_1(p(|A_\xi|\,;\,\tau))\sqrt{E|A_\xi|^2} \end{aligned} \tag{85}$$

In particular, supposing that all the functions d_1, d_2 and $\bar{\lambda}$ do not depend on time τ, eq. (85) becomes

$$(d_1 + d_2\bar{\lambda})\sup_{T}\sqrt{E|A_\xi|^2} \geqq ES_V(\omega,\,h,\,\tau_d) \doteqdot E\sup_{T}|A_\xi| \geqq d_1\sup_{T}\sqrt{E|A_\xi|^2} \tag{86}$$

Hence there exists a positive number $\tilde{\lambda}$ which satisfies the following equation:

$$ES_V(\omega,\,h,\,\tau_d) = (d_1 + \tilde{\lambda}d_2)\sup_{T}\sqrt{E|A_\xi|^2}, \qquad \bar{\lambda} \geqq \tilde{\lambda} \geqq 0 \tag{87}$$

Strictly speaking, for the quasi-stationary random process, which is a kind of non-stationary random process, neither d_1 nor d_2 are independent of time τ. Substituting eqs. (75)~(77) in each equation of (84), d_1 and d_2 are obtained as the even functions of the dimensionless quantity ξ as follows:

$$\begin{aligned} d_1(p(|A_\xi|\,;\,\tau)) &= \frac{\sqrt{\pi}}{2}\sqrt{\frac{1+\xi^2}{1+\xi\tan^{-1}\xi}} = d_1(\xi) \\ d_2(p(|A_\xi|\,;\,\tau)) &= \sqrt{\frac{1-\frac{\pi}{4}+\xi\left(\tan^{-1}\xi - \frac{\pi}{4}\,\xi\right)}{1+\xi\tan^{-1}\xi}} = d_2(\xi) \end{aligned} \tag{88}$$

From the above equations the ratio of d_2 to d_1 is given by

$$\sigma(p(|A_\xi|\,;\tau))=\frac{\sqrt{V|A_\xi|}}{E|A_\xi|}=\frac{\sqrt{V(\zeta)}}{E(\zeta)}=\frac{d_2(\xi)}{d_1(\xi)}=\sqrt{\frac{\frac{4-\pi}{\pi}+\xi\left(\frac{4}{\pi}\tan^{-1}\xi-\xi\right)}{1+\xi^2}}=\sigma(\xi) \quad (89)$$

Therefore, for the validity of eq. (87) it is necessary that the dimensionless quantity ξ defined by eq. (71) be independent of time τ.

In connection with the validity of eq. (87), the behaviour of the time-function ξ is examined in the following, for the output response of the linear system, which has the impulsive response $g(\tau)=\exp(-h|\omega|\tau)\sin\omega\tau$, to quasi-stationary random excitations.

In general, for an arbitrary time-invariant linear system subjected to the quasi-stationary random excitations defined by eq. (22), each element of the co-variance matrix associated with the output response J and its time derivative $\dot{J}$ is expressed as follows:[8)]

$$K_{JJ}(\tau_1,\tau_2)=C(J_\xi(\tau_1),J_\xi(\tau_2))=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(\omega';\tau_1)S_\phi(\omega')X^*(\omega';\tau_2)d\omega' \quad (90)$$

$$\begin{aligned}K_{J\dot{J}}(\tau_1,\tau_2)&=C(J_\xi(\tau_1),J_{\xi\tau}{}^{(1)}(\tau_2))=K_{JJ\tau_2}{}^{(1)}(\tau_1,\tau_2)\\&=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(\omega';\tau_1)S_\phi(\omega')X_\tau{}^{(1)*}(\omega';\tau_2)d\omega'\end{aligned} \quad (91)$$

$$\begin{aligned}K_{\dot{J}J}(\tau_1,\tau_2)&=C(J_{\xi\tau}{}^{(1)}(\tau_1),J_\xi(\tau_2))=K_{JJ\tau_1}{}^{(1)}(\tau_1,\tau_2)\\&=\frac{1}{2\pi}\int_{-\infty}^{\infty}X_\tau{}^{(1)}(\omega';\tau_1)S_\phi(\omega')X^*(\omega';\tau_2)d\omega'\end{aligned} \quad (92)$$

$$\begin{aligned}K_{\dot{J}\dot{J}}(\tau_1,\tau_2)&=C(J_{\xi\tau}{}^{(1)}(\tau_1),J_{\xi\tau}{}^{(1)}(\tau_2))=K_{JJ\tau_1}{}^{(1)}{}_{\tau_2}{}^{(1)}(\tau_1,\tau_2)\\&=\frac{1}{2\pi}\int_{-\infty}^{\infty}X_\tau{}^{(1)}(\omega';\tau_1)S_\phi(\omega')X_\tau{}^{(1)*}(\omega';\tau_2)d\omega'\end{aligned} \quad (93)$$

where

$$X(\omega';\tau)=G(\omega';\tau)\exp(j\omega'\tau) \quad (94)$$

$$\begin{aligned}G(\omega';\tau)&=\int_{-\infty}^{\tau}D(\mu;R^1{}_{0\tau_d})a(\mu)g(\tau-\mu)\exp\{-j(\tau-\mu)\omega'\}d\mu\\&=\int_0^{\tau_m}a(\mu)g(\tau-\mu)\exp\{-j(\tau-\mu)\omega'\}d\mu\end{aligned} \quad (95)$$

in which $g(\tau)$ is the impulsive response of the linear system, ${}_\tau{}^{(1)}$ or ${}_{\tau_i}{}^{(1)}$ denotes the partial differentiation with respect to τ or τ_i and the superscript $*$ means the complex-conjugate.

Substituting $\tau_1=\tau_2=\tau$ in eqs. (90)~(93), the variances of J and $\dot{J}$ and the co-variance between J and $\dot{J}$ at time τ are obtained as follows:

$$K_{JJ}=K_{JJ}(\tau,\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty}|X(\omega';\tau)|^2S_\phi(\omega')d\omega' \quad (96)$$

$$\begin{aligned}K_{J\dot{J}}=K_{\dot{J}J}=K_{J\dot{J}}(\tau,\tau)=K_{\dot{J}J}(\tau,\tau)&=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(\omega';\tau)X_\tau{}^{(1)*}(\omega';\tau)S_\phi(\omega')d\omega'\\&=\frac{1}{2\pi}\int_{-\infty}^{\infty}R(X(\omega';\tau)X_\tau{}^{(1)*}(\omega';\tau))S_\phi(\omega')d\omega'\end{aligned} \quad (97)$$

$$K_{\dot{J}\dot{J}}=K_{\dot{J}\dot{J}}(\tau,\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty}|X_\tau{}^{(1)}(\omega';\tau)|^2S_\phi(\omega')d\omega' \quad (98)$$

Substituting $g(\tau)=\exp(-h|\omega|\tau)\sin\omega\tau$ in eq. (95), eq. (94) and its time derivative are expressed in terms of the notations given by eqs. (43) and (44) as follows:

$$X(\omega';\tau)=-\frac{1}{2}\{\exp(j\omega\tau)c(\omega',\tau;\omega,h,\tau_d)\\ -\exp(-j\omega\tau)c^*(-\omega',\tau;\omega,h,\tau_d)\} \tag{99}$$

$$X_\tau^{(1)}(\omega';\tau)=\frac{\omega}{2}\{\exp(j\omega\tau)c(\omega',\tau;\omega,h,\tau_d)\\ +\exp(-j\omega\tau)c^*(-\omega',\tau;\omega,h,\tau_d)\}\\ -\frac{j}{2}\{\exp(j\omega\tau)c_\tau^{(1)}(\omega',\tau;\omega,h,\tau_d)\\ -\exp(-j\omega\tau)c_\tau^{(1)*}(-\omega',\tau;\omega,h,\tau_d)\} \tag{100}$$

where

$$c_\tau^{(1)}(\omega',\tau;\omega,h,\tau_d)=-h|\omega|\exp(-h|\omega|\tau)B(\omega';\omega,h,\tau)\\ +a(\tau)\exp\{-j(\omega-\omega')\tau\}, \qquad 0\leqq\tau<\tau_d\\ =-h|\omega|c(\omega',\tau;\omega,h,\tau_d)+a(\tau)\exp\{-j(\omega-\omega')\tau\} \tag{101}$$

$$c_\tau^{(1)}(\omega',\tau;\omega,h,\tau_d)=-h|\omega|\exp(-h|\omega|\tau)B(\omega';\omega,h,\tau_d)\\ =-h|\omega|c(\omega',\tau;\omega,h,\tau_d), \qquad \tau>\tau_d \tag{102}$$

and

$$c(-\omega',\tau;\omega,h,\tau_d)=c^*(\omega',\tau;-\omega,h,\tau_d) \tag{103}$$

By expressing each integrand in eqs. (96)~(98) in terms of the notations defined by eqs. (43), (101) and (102) and by taking into account the relations,

$$S_\psi^*(\omega')=S_\psi(\omega'), \qquad S_\psi(\omega')=S_\psi(-\omega') \tag{104}$$

eqs. (96)~(98) are written in the following forms respectively:

$$K_{JJ}=\frac{1}{2\pi}\int_{-\infty}^{\infty}Y_{JJ}(\omega';\tau)S_\psi(\omega')d\omega' \tag{105}$$

$$Y_{JJ}(\omega';\tau)=\frac{1}{2}[|c(\omega',\tau;\omega,h,\tau_d)|^2\\ -\boldsymbol{R}\{\exp(2j\omega\tau)c(\omega',\tau;\omega,h,\tau_d)c(-\omega',\tau;\omega,h,\tau_d)\}] \tag{106}$$

$$K_{J\dot{J}}=K_{\dot{J}J}=\frac{1}{2\pi}\int_{-\infty}^{\infty}Y_{J\dot{J}}(\omega';\tau)S_\psi(\omega')d\omega' \tag{107}$$

$$Y_{J\dot{J}}(\omega';\tau)=-\frac{h|\omega|}{2}|c(\omega',\tau;\omega,h,\tau_d)|^2\\ +\frac{h|\omega|}{2}\boldsymbol{R}\{\exp(2j\omega\tau)c(\omega',\tau;\omega,h,\tau_d)c(-\omega',\tau;\omega,h,\tau_d)\}\\ +\frac{\omega}{2}\boldsymbol{I}\{\exp(2j\omega\tau)c(\omega',\tau;\omega,h,\tau_d)c(-\omega',\tau;\omega,h,\tau_d)\} \tag{108}$$

and

$$K_{\dot{J}\dot{J}}=\frac{1}{2\pi}\int_{-\infty}^{\infty}Y_{\dot{J}\dot{J}}(\omega';\tau)S_\psi(\omega')d\omega' \tag{109}$$

$$
\begin{aligned}
Y_{JJ}(\omega' ; \tau) = & \frac{1}{2}\omega^2(1+h^2)|c(\omega', \tau ; \omega, h, \tau_d)|^2 \\
& + \frac{1}{2}\omega^2(1-h^2)\boldsymbol{R}\{\exp(2j\omega\tau)c(\omega', \tau ; \omega, h, \tau_d)c(-\omega', \tau ; \omega, h, \tau_d)\} \\
& - h|\omega|\omega\boldsymbol{I}\{\exp(2j\omega\tau)c(\omega', \tau ; \omega, h, \tau_d)c(-\omega', \tau ; \omega, h, \tau_d)\}
\end{aligned} \tag{110}
$$

In particular, in the case where the envelope of the quasi-stationary random process $a(\tau)=1$, eqs. (43), (101) and (102) reduce to the following forms respectively:

$$
c(\omega', \tau ; \omega, h, \tau_d) = \exp(-h|\omega|\tau)\frac{\exp\{(h|\omega|-j(\omega-\omega'))\tau_m\}-1}{h|\omega|-j(\omega-\omega')} \tag{111}
$$

$$
\begin{aligned}
c_\tau^{(1)}(\omega', \tau ; \omega, h, \tau_d) = & -h|\omega|\exp(-h|\omega|\tau)\frac{\exp\{(h|\omega|-j(\omega-\omega'))\tau\}-1}{h|\omega|-j(\omega-\omega')} \\
& + \exp\{-j(\omega-\omega')\tau\}, \qquad 0 \leqq \tau < \tau_d
\end{aligned} \tag{112}
$$

$$
c_\tau^{(1)}(\omega', \tau ; \omega, h, \tau_d) = -h|\omega|\exp(-h|\omega|\tau)\frac{\exp\{(h|\omega|-j(\omega-\omega'))\tau_d\}-1}{h|\omega|-j(\omega-\omega')}, \qquad \tau > \tau_d \tag{113}
$$

In another special case where the damping parameter $h=0$, eqs. (43), (101) and (102) become respectively:

$$
\begin{aligned}
c(\omega', \tau ; \omega, 0, \tau_d) & = \int_0^{\tau_m} a(\mu)\exp\{-j(\omega-\omega')\mu\}d\mu \\
& = \frac{1}{2\pi}\int_{-\infty}^{\infty} A(j\nu)\frac{1-\exp\{-j(\omega-\omega'-\nu)\tau_m\}}{j(\omega-\omega'-\nu)}d\nu
\end{aligned} \tag{114}
$$

$$
c_\tau^{(1)}(\omega', \tau ; \omega, 0, \tau_d) = a(\tau)\exp\{-j(\omega-\omega')\tau\}, \qquad 0 \leqq \tau < \tau_d \tag{115}
$$

$$
c_\tau^{(1)}(\omega', \tau ; \omega, 0, \tau_d) = 0, \qquad \tau > \tau_d \tag{116}
$$

Since the non-stationary character of the output response seems to be remarkable in the case where $a(\tau)=1$ and $h=0$, the behaviour of ξ as a function of time is examined for this special case. In this case eqs. (105), (107) and (109) are evaluated respectively as:

$$
K_{JJ} = \frac{1}{2\pi}\frac{1-\cos\omega\tau_m}{\omega^2}*S_\phi(\omega) - \frac{\cos(2\tau-\tau_m)}{\pi}\int_{-\infty}^{\infty}\frac{\sin\frac{\omega+\omega'}{2}\tau_m \sin\frac{\omega-\omega'}{2}\tau_m}{\omega^2-\omega'^2}d\omega' \tag{117}
$$

$$
K_{JJ} = \frac{\omega\sin(2\tau-\tau_m)}{\pi}\int_{-\infty}^{\infty}\frac{\sin\frac{\omega+\omega'}{2}\tau_m \sin\frac{\omega-\omega'}{2}\tau_m}{\omega^2-\omega'^2}d\omega' \tag{118}
$$

$$
K_{JJ} = \frac{\omega^2}{2\pi}\left(\frac{1-\cos\omega\tau_m}{\omega^2}*S_\phi(\omega)\right) + \frac{\omega^2\cos(2\tau-\tau_m)}{\pi}\int_{-\infty}^{\infty}\frac{\sin\frac{\omega+\omega'}{2}\tau_m \sin\frac{\omega-\omega'}{2}\tau_m}{\omega^2-\omega'^2}d\omega' \tag{119}
$$

in which the symbol $*$ indicates the convolution integral defined in the full frequency domain $(-\infty, \infty)$.

By making use of the limiting formulae,

$$
\lim_{\tau_m\to\infty}\frac{1}{\pi}\frac{1-\cos\omega\tau_m}{\omega^2} = \tau_m\delta(\omega) \tag{120}
$$

$$\lim_{\tau_m\to\infty}\frac{1}{\pi}\frac{\sin\frac{\omega+\omega'}{2}\tau_m\sin\frac{\omega-\omega'}{2}\tau_m}{\omega^2-\omega'^2}=\frac{\sin\omega\tau_m}{4\omega}\{\delta(\omega+\omega')+\delta(\omega-\omega')\} \tag{121}$$

in which $\delta(\omega)$ denotes Dirac's delta-function, eqs. (117), (118) and (119) can be approximately expressed as follows for a sufficiently large τ_m:

$$K_{JJ}=\left(\frac{\tau_m}{2}-\frac{\cos\omega(2\tau-\tau_m)\sin\omega\tau_m}{2\omega}\right)S_\phi(\omega) \tag{122}$$

$$K_{J\dot{J}}=\frac{\sin\omega(2\tau-\tau_m)\sin\omega\tau_m}{2}S_\phi(\omega) \tag{123}$$

$$K_{\dot{J}\dot{J}}=\left(\frac{\tau_m}{2}+\frac{\cos\omega(2\tau-\tau_m)\sin\omega\tau_m}{2\omega}\right)\omega^2S_\phi(\omega) \tag{124}$$

Substituting eqs. (122)$\sim$(124) in the first equation of (68) and eq. (71) the normalized co-variance ρ and the dimensionless quantity ξ are determined respectively as:

$$\rho=\frac{\sin\omega(2\tau-\tau_m)\sin\omega\tau_m}{\sqrt{\omega^2\tau_m^2-\cos^2\omega(2\tau-\tau_m)\sin^2\omega\tau_m}} \tag{125}$$

$$\xi=\frac{\sin\omega(2\tau-\tau_m)\sin\omega\tau_m}{\sqrt{\omega^2\tau_m^2-\sin^2\omega\tau_m}},\quad \tau_m=\min(\tau,\tau_d) \tag{126}$$

From the above equations it is found that for the time domain $[0, \tau_d]$ both ρ and ξ are non-negative hence the random process is divergent, on the other hand, for the time domain (τ_d, ∞) both ρ and ξ sinusoidally oscillate hence the random process is stationary in a sense. Also it is found that if τ_m becomes sufficiently large, both ρ and ξ decreases proportionally to τ_m^{-1}.

For the special case where the envelope $a(\tau)$ of the quasi-stationary random excitations is the step function and the stationary random process $\phi(\tau)$ is Gaussian and white, the quantities ρ and ξ can be determined by solving the Fokker-Planck equation. The impulsive response and the transfer function of the linear dynamic system considered are given by

$$g(\tau)=\exp(-h|\omega|\tau)\sin\omega\tau \tag{127}$$

$$G(s)=\frac{\omega}{s^2+2h|\omega|s+(1+h^2)\omega^2} \tag{128}$$

Hence the differential equation governing the output response J of the dynamic system subjected to the input excitation $s(\tau)\phi(\tau)$ becomes

$$\left(\frac{d^2}{d\tau^2}+2h|\omega|\frac{d}{d\tau}+(1+h^2)\omega^2\right)J=\omega s(\tau)\phi(\tau) \tag{129}$$

in which $s(\tau)$ is the step-function and $\phi(\tau)$ is the Gaussian stationary random process with the white power spectral density $S_\phi(\omega)=c^2$.

Transforming the variables J and $\dot{J}=d/d\tau J$ into the new variables y_1 and y_2 by the equation,

$$\begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix}=\begin{bmatrix} -\lambda_2 & 1 \\ -\lambda_1 & 1 \end{bmatrix}\begin{Bmatrix} J \\ \dot{J} \end{Bmatrix} \tag{130}$$

where

$$\begin{matrix}\lambda_1\\ \lambda_2\end{matrix}=(-h\pm j)\omega \tag{131}$$

the Fokker-Planck equation governing the joint probability density distribution $f(y_1, y_2, ; \tau)$ is obtained as follows:

$$\frac{\partial f}{\partial \tau}=-\lambda_1\frac{\partial}{\partial y_1}(y_1 f)-\lambda_2\frac{\partial}{\partial y_2}(y_2 f)+\frac{c^2\omega^2}{2}\left(\frac{\partial}{\partial y_1}+\frac{\partial}{\partial y_2}\right)^2 f \tag{132}$$

Solving the above partial differential equation by means of the Fourier transform technique under the zero initial condition, the co-variance matrix associated with J and $\dot{J}$ is determined as follows:[21]

$$\begin{bmatrix} K_{JJ} & K_{J\dot{J}} \\ K_{\dot{J}J} & K_{\dot{J}\dot{J}} \end{bmatrix}=\frac{c^2}{4}\begin{bmatrix} 1 & -1 \\ \lambda_1 & -\lambda_2 \end{bmatrix}\begin{bmatrix} \frac{1}{2\lambda_1}(1-\exp(2\lambda_1\tau)) & \frac{1}{\lambda_1+\lambda_2}(1-\exp(\lambda_1+\lambda_2)\tau) \\ \frac{1}{\lambda_1+\lambda_2}(1-\exp(\lambda_1+\lambda_2)\tau) & \frac{1}{2\lambda_2}(1-\exp(2\lambda_2\tau)) \end{bmatrix}\begin{bmatrix} 1 & \lambda_1 \\ -1 & \lambda_2 \end{bmatrix} \tag{133}$$

Substituting eq. (131) in eq. (133) each element of the co-variance matrix is expressed as

$$K_{JJ}=\frac{c^2}{4h(1+h^2)|\omega|}\Big[1-(1+h^2)\exp(-2h|\omega|\tau)\cdot\Big\{1-\frac{h^2}{1+h^2}\cos 2\omega\tau+\frac{h}{1+h^2}\sin 2|\omega|\tau\Big\}\Big] \tag{134-a}$$

$$K_{J\dot{J}}=K_{\dot{J}J}=\frac{c^2}{2}\exp(-2h|\omega|\tau)\sin^2\omega\tau \tag{134-b}$$

$$K_{\dot{J}\dot{J}}=\frac{c^2|\omega|}{4h}\Big[1-(1+h^2)\exp(-2h|\omega|\tau)\cdot\Big\{1-\frac{h^2}{1+h^2}\cos 2\omega\tau-\frac{h}{1+h^2}\sin 2|\omega|\tau\Big\}\Big] \tag{134-c}$$

By making use of the above equations, ρ and ξ, which are defined by the first equation of (68) and eq. (71) respectively, are determined as follows:

$$\rho=-\frac{2h\sqrt{1+h^2}\exp(-2h|\omega|\tau)\sin^2\omega\tau}{\sqrt{1+\exp(-4h|\omega|\tau)-2(1+h^2)\exp(-2h|\omega|\tau)\left(1-\frac{h^2}{1+h^2}\cos 2\omega\tau\right)+4h^2(1+h^2)\exp(-4h|\omega|\tau)\sin^4\omega\tau}}\quad \text{for}\quad h>0 \tag{135}$$

$$\xi=\frac{2h\sqrt{1+h^2}\exp(-2h|\omega|\tau)\sin^2\omega\tau}{\sqrt{1+\exp(-4h|\omega|\tau)-2(1+h^2)\exp(-2h|\omega|\tau)\left(1-\frac{h^2}{1+h^2}\cos 2\omega\tau\right)}}\quad \text{for}\quad h>0 \tag{136}$$

In particular, in the case where the damping parameter $h=0$, ρ and ξ are determined as the limiting values of eqs. (135) and (136) when $h\to 0$ and are respectively given as

$$\rho=-\frac{\sin^2\omega\tau}{\sqrt{\omega^2\tau^2-\cos^2\omega\tau\sin^2\omega\tau}}\quad \text{for}\quad h=0 \tag{137}$$

$$\xi=\frac{\sin^2\omega\tau}{\sqrt{\omega^2\tau^2-\sin^2\omega\tau}}\quad \text{for}\quad h=0 \tag{138}$$

In the time domain $[0, \tau_d]$ the above equations coincide with eqs. (125) and (126) which are obtained for a sufficiently large $\tau_m=\min(\tau, \tau_d)$. Then it is suggested that in the case where $h=0$, eqs. (125) and (126) are approximately valid in the full time domain $[0, \infty)$ if the envelope $a(\tau)$ of the quasi-stationary random excitations is a slowly varying time-function and the power spectral density of the stationary random process $\psi(\tau)$ is comparatively flat. Also from eqs. (135)~(138), it is suggested that as far as the envelope $a(\tau)$ is a slowly varying time-function, both ρ and ξ decrease exponentially for the case where $h>0$, while they decrease in the order of τ_m^{-1} for the case where $h=0$ as τ_m increases.

As mentioned above, if the envelope $a(\tau)$ of the quasi-stationary random excitations is a weak function of time and τ_m is sufficiently large, quantities ρ and ξ may both be approximately regarded as zero where the damping parameter h is zero or not. Hence, if the maximum response of the linear system subjected to the quasi-stationary Gaussian random process occurs in such circumstances, eqs. (86) and (87) are approximately valid. Even if the envelope $a(\tau)$ is a strong function of time, ρ and ξ may be approximately regarded as zero at the time τ_{max} when the maximum response occurs, because τ_{max} is a kind of stationary point of ρ or ξ from the macroscopic point of view. Hence it is found that the mean value of response spectra and their upper and lower bounds can be expressed in the forms given by eq. (87) and both sides of eq. (86), respectively.

Substituting $\rho=\xi=0$ in eqs. (67) and (70) the probability density function of the envelope of output response and its non-dimensional expression are obtained respectively as:

$$p(|A_\xi|)=\frac{|A_\xi|}{K_{JJ}}\exp\left(-\frac{|A_\xi|^2}{2K_{JJ}}\right) \tag{139}$$

and

$$\bar{p}(\zeta)=\zeta\exp\left(-\frac{\zeta^2}{2}\right) \tag{140}$$

both of which are called the Rayleigh distribution. By substituting $\xi=0$ in eq. (88) the quantities d_1 and d_2 are reduced to the following constants which are characteristic values associated with the Rayleigh distribution:

$$d_1=\frac{E|A_\xi|}{\sqrt{E|A_\xi|^2}}=\frac{\sqrt{\pi}}{2},\qquad d_2=\frac{\sqrt{V|A_\xi|}}{\sqrt{E|A_\xi|^2}}=\sqrt{1-\frac{\pi}{4}} \tag{141}$$

On the other hand, the maximum normalized random variable $\bar{\lambda}$ referred to $|A_\xi|$ and the equivalent coefficient $\tilde{\lambda}$ which gives the mean value of the response spectra of the quasi-stationary random excitation mainly depend on the boundedness of the probability distribution of the stationary random process $\psi(\tau)$. The numerical values of $\bar{\lambda}$ and $\tilde{\lambda}$ may be roughly estimated from the Rayleigh distribution by supposing the exceeding probability, for instance, $\bar{\lambda}=\tilde{\lambda}=3.05$ is obtained corresponding to the value of the exceeding probability 5×10^{-3}. However, the values of $\bar{\lambda}$ and $\tilde{\lambda}$ seem to be affected not only by the bounded probability distribution of the stationary random process $\psi(\tau)$ but also by the system parameters ω, h and the various parameters describing the envelope $D(\tau\,;\,R^{1}{}_{0\tau_d})a(\tau)$ of the quasi-stationary random process.

Since it is difficult to analytically determine the functional forms of $\bar{\lambda}$ and $\tilde{\lambda}$ at the present stage, the following semi-experimental method of evaluating the values of $\bar{\lambda}$ and $\tilde{\lambda}$ is adopted:

First, the energy spectral density $E|A_\xi(\tau\,;\,\omega, h, \tau_d)|^2$ of the modified quasi-stationary random process is formally expressed as the product of the non-negative function of ω only and that of ω, h and τ_d as follows:

$$\begin{aligned} S_{E\xi}(\tau\,;\,\omega, h, \tau_d) &= E|A_\xi(\tau\,;\,\omega, h, \tau_d)|^2, \qquad S_s(\omega)\geqq 0 \\ &= D(\tau\,;\,\omega, h, \tau_d)S_s(\omega), \qquad D(\tau\,;\,\omega, h, \tau_d)\geqq 0 \end{aligned} \tag{142}$$

Then defining the random time-functions by the equations,

$$J_s(\tau\,;\,\omega, h, \tau_d) = \frac{J_\xi(\tau\,;\,\omega, h, \tau_d)}{\sqrt{D(\tau\,;\,\omega, h, \tau_d)}} \tag{143}$$

and

$$A_s(\tau\,;\,\omega, h, \tau_d) = \frac{A_\xi(\tau\,;\,\omega, h, \tau_d)}{\sqrt{D(\tau\,;\,\omega, h, \tau_d)}} \tag{144}$$

the following relations are obtained by making use of eqs. (24) and (142):

$$J_s(\tau\,;\,\omega, h, \tau_d) = |A_s(\tau\,;\,\omega, h, \tau_d)|\sin(\omega\tau + \arg A_\xi(\tau\,;\,\omega, h, \tau_d)) \tag{145}$$

$$E|A_s(\tau\,;\,\omega, h, \tau_d)|^2 = S_s(\omega) \tag{146}$$

In the sense that the square mean $E|A_s|^2$ does not depend on time τ, the random process defined by eq. (144) is considered as the almost stationary random process, hence hereafter the random function $|A_s|$ is called the pseudo-stationary random function associated with $|A_\xi|$.

From eqs. (66) and (144) the smaller upper bound of the mean value of the response spectra than in eq. (83) is obtained as follows:

$$\begin{aligned} ES_V(\omega, h, \tau_d) &\doteqdot E\sup_T|A_\xi(\tau\,;\,\omega, h, \tau_d)| \\ &\leqq \sup_T\sqrt{D(\tau\,;\,\omega, h, \tau_d)}\,E\sup_T|A_s(\tau\,;\,\omega, h, \tau_d)| \end{aligned} \tag{147}$$

Assuming the pseudo-stationary random process $|A_s|$ to be ergodic and stationary the right-hand side of eq. (147) may be expressed as

$$\begin{aligned} &\sup_T\sqrt{D(\tau\,;\,\omega, h, \tau_d)}\,E_E\sup_T|A_s(\tau\,;\,\omega, h, \tau_d)| \\ &\quad = \sup_T\sqrt{D(\tau\,;\,\omega, h, \tau_d)}\,E_T\sup_E|A_s(\tau\,;\,\omega, h, \tau_d)| \end{aligned} \tag{148}$$

in which subscripts E and T denote the averaging operators with respect to ensemble and time, respectively.

By applying the averaging operators to the similar expressions associated with $|A_s|$ as in eq. (80) and by taking into consideration eq. (82) and the ergodicity of $|A_s|$, the following equation is obtained;

$$\begin{aligned} E_E\sup_T|A_s| &= E_T|A_s| + \bar{\lambda}_{av}\sqrt{V_T|A_s|} \\ &= E_T\sup_E|A_s| = E_E|A_s| + \bar{\lambda}_{av}\sqrt{V_E|A_s|} \end{aligned} \tag{149}$$

where

$$\bar{\lambda}_{av} = E_E\bar{\lambda}_T = E_T\bar{\lambda}_E > 1 \tag{150}$$

Hence eq. (147) can be rewritten as follows:

$$\begin{aligned} ES_V(\omega, h, \tau_d) &\leqq \sup_T \sqrt{D(\tau;\omega, h, \tau_d)}\{E_T|A_s| + \bar{\lambda}_{av}\sqrt{V_T|A_s|}\} \\ &= \sup_T \sqrt{D(\tau;\omega, h, \tau_d)}\{E_E|A_s| + \bar{\lambda}_{av}\sqrt{V_E|A_s|}\} \end{aligned} \tag{151}$$

Since both quantities ρ and ξ may be regarded as zero for the pseudo-stationary random process $|A_s|$, the Rayleigh distribution given by eq. (139) or (140) can be applied to the amplitude probability density distribution of $|A_s|$. Hence by making use of eq. (141), the upper bound of the mean value of response spectra given by eq. (151) is expressed as follows;

$$\begin{aligned} ES_V(\omega, h, \tau_d) &\leqq \sup_T \sqrt{D(\tau;\omega, h, \tau_d)}\left(\frac{\sqrt{\pi}}{2} + \bar{\lambda}_{av}\sqrt{1-\frac{\pi}{4}}\right)\sqrt{E_E|A_s|^2} \\ &= \left(\frac{\sqrt{\pi}}{2} + \bar{\lambda}_{av}\sqrt{1-\frac{\pi}{4}}\right)\sqrt{\sup_T E_E|A_\xi|^2} \\ &= \left(\frac{\sqrt{\pi}}{2} + \bar{\lambda}_{av}\sqrt{1-\frac{\pi}{4}}\right)\sqrt{\sup_T S_{E\xi}(\tau;\omega, h, \tau_d)} \end{aligned} \tag{152}$$

similarly, the lower bound of the mean value of response spectra is obtatined as follows:

$$\begin{aligned} ES_V(\omega, h, \tau_d) &\geqq \sup_T \sqrt{D(\tau;\omega, h, \tau_d)}\frac{\sqrt{\pi}}{2}\sqrt{E_E|A_s|^2} \\ &= \frac{\sqrt{\pi}}{2}\sqrt{\sup_T S_{E\xi}(\tau;\omega, h, \tau_d)} \end{aligned} \tag{153}$$

Therefore there exists the quantity λ, so that the mean value of the response spectra of the quasi-stationary random process is expressed as

$$\begin{aligned} ES_V(\omega, h, \tau_d) &= \left(\frac{\sqrt{\pi}}{2} + \lambda\sqrt{1-\frac{\pi}{4}}\right)\sqrt{\sup_T S_{E\xi}(\tau;\omega, h, \tau_d)} \\ &\bar{\lambda}_{av} \geqq \lambda \geqq 0 \end{aligned} \tag{154}$$

In general, both quantities $\bar{\lambda}_{av}$ and λ are the positive functions of ω, h and τ_d. $\bar{\lambda}_{av}$ is not substantially smaller than λ. However, if $D(\tau;\omega, h, \tau_d)$ is a slowly varying time-function compared with the fluctuation of $|A_s|$, $\bar{\lambda}_{av}$ may be approximately regarded as the least upper bound of λ, that is,

$$\lambda(\omega, h, \tau_d) \doteqdot \bar{\lambda}_{av}(\omega, h, \tau_d) \tag{155}$$

because the following approximate equation is valid:

$$\begin{aligned} \sup_T|A_\xi| &\doteqdot \sup_T\sqrt{D}\sup_T|A_s| \\ &= \sup_T\sqrt{D}(E_T|A_s| + \bar{\lambda}_T\sqrt{V_T|A_s|}) \end{aligned} \tag{156}$$

Eqs. (155) and (156) may be valid in the case where the envelope $a(\tau)D(\tau; R'_{0\tau_d})$ of the quasi-stationary random excitations is a slowly varying time-function having sufficiently large duration time τ_d and the damping parameter h is positive, even if it is only slightly positive. On the contrary, in the case where the envelope is a strong function of time or the damping parameter is zero, λ may be considerably smaller than $\bar{\lambda}_{av}$.

In the following, the upper and lower limits of the response spectra of the quasi-stationary random excitations are considered in the case where eq. (156) is valid.

The upper and lower limits of the response spectra having the bounded probability distribution may be expressed respectively as;

$$\begin{aligned} \sup_E S_V &= E_E S_V + \bar{\mu}\sqrt{V_E S_V} \\ \inf_E S_V &= E_E S_V + \underline{\mu}\sqrt{V_E S_V} \end{aligned} \tag{157}$$

in which $\bar{\mu}$ and $\underline{\mu}$ are the maximum and minimum values of the normalized random variable associated with S_V, respectively.

The variance of the response spectra $V_E S_V$ contained in eq. (157) can be approximately expressed in the following form similar to eq. (66):

$$\begin{aligned} V_E S_V &= E_E(S_V^2) - (E_E S_V)^2 \\ &\doteqdot E_E(\sup_T |A_\xi|)^2 - (E_E \sup_T |A_\xi|)^2 \end{aligned} \tag{158}$$

Substituting eq. (156) in eq. (158) and by taking into consideration the ergodic property of the pseudo-stationary random process $|A_s|$ the variance of S_V is written as

$$\begin{aligned} V_E S_V &= \sup DV\bar{\lambda}V|A_s| = V\bar{\lambda}\Big(1-\frac{\pi}{4}\Big)\sup_T DE|A_s|^2 \\ &= \Big(1-\frac{\pi}{4}\Big)V\bar{\lambda}\sup_T E|A_\xi|^2 = \Big(1-\frac{\pi}{4}\Big)V\bar{\lambda}\sup_T S_{E\xi} \end{aligned} \tag{159}$$

Hence by making use of eqs. (150), (154), (157) and (159), the upper and lower limits of the response spectra of the quasi-stationary random process are expressed as follows;

$$\sup_E S_V = \Big(\frac{\sqrt{\pi}}{2} + \sqrt{1-\frac{\pi}{4}}\,(\bar{\lambda}_{av} + \bar{\mu}\sqrt{V\bar{\lambda}})\Big)\sqrt{\sup_T S_{E\xi}} \tag{160}$$

$$\inf_E S_V = \Big(\frac{\sqrt{\pi}}{2} + \sqrt{1-\frac{\pi}{4}}\,(\bar{\lambda}_{av} + \underline{\mu}\sqrt{V\bar{\lambda}})\Big)\sqrt{\sup_T S_{E\xi}} \tag{161}$$

in which

$$\bar{\mu} = \frac{\sup_E S_V - E_E S_V}{\sqrt{V_E S_V}} = \frac{\sup\bar{\lambda} - E\bar{\lambda}}{\sqrt{V\bar{\lambda}}} > 0 \tag{162}$$

$$\underline{\mu} = \frac{\inf_E S_V - E_E S_V}{\sqrt{V_E S_V}} = \frac{\inf\bar{\lambda} - E\bar{\lambda}}{\sqrt{V\bar{\lambda}}} < 0 \tag{163}$$

As shown in eqs. (162) and (163), the quantities $\bar{\mu}$ and $\underline{\mu}$ which have been introduced in eq. (157) as the maximum and minimum values of the normalized random variable associated with S_V are also considered as the maximum and minimum values of the normalized random vaiable $\bar{\lambda}$ associated with the pseudo-stationary random process $|A_s|$ defined by

$$\bar{\lambda}=\frac{\sup|A_s|-E|A_s|}{\sqrt{V|A_s|}} \tag{164}$$

In general, the quantities $\bar{\mu}$ and $\underline{\mu}$ depend on the statistical properties of the original quasi-stationary random process $f(\tau)=a(\tau)\psi(\tau)$ as well as the parameters ω, h and τ_d.

As shown in eqs. (154), (155), (160) and (161) the mean value and the upper and lower limits of the response spectra of the quasi-stationary random excitations having finite duration time are obtained as the products of the root of the maximum value of energy spectral density of the modified quasi-stationary random process and the relevant multiplication factors, which are expressed in terms of the characteristic values of the Rayleigh distribution and the probability distribution of the maximum value of the normalized random variable associated with the pseudo-stationary random process. In eqs. (155), (160) and (161), the quantities $\bar{\lambda}_{av}=E\bar{\lambda}$, $V\bar{\lambda}$, $\bar{\mu}$ and $\underline{\mu}$ are the deterministic functions of ω, h and τ_d which are explicitly concerned with the probability distribution of the bounded random variable $\bar{\lambda}$ defined by eq. (164). The quantity λ which appears in eq. (154) is also the deterministic function of ω, h and τ_d and is approximately equal to $\bar{\lambda}_{av}$ under certain conditions previously mentioned. However, it is generally the equivalent coefficient associated with the Rayleigh distribution which is determined so as to give precisely the mean value of response spectra as shown in eq. (154) even in the case where $\bar{\lambda}_{av}$ gives an upper bound of response spectra or the probability distribution of $|A_s|$ deviating from the Rayleigh distribution.

Since it seems to be difficult to analytically determine the functional forms of these quantities their definite expressions must be determined experimentally by using the so-called simulation method, that is, based upon the results of numerical analyses of the responses of a single-degree-of-freedom, linear system subjected to the quasi-stationary random excitations which are appropriately generated by making use of a simulation procedure. In this simulation method for estimating the functional forms of the quantities $\bar{\lambda}_{av}$, $V\lambda$, $\bar{\mu}$, $\underline{\mu}$ and λ the assumed ergodic property of the pseudo-stationary random process $|A_s|$ may be conveniently used to obtain the required data based upon a rather small number of sample functions of the simulated random processes.

On the other hand, the maximum value of the energy spectral density $\sup_{T} S_{E\xi}$ of the modified quasi-stationary random process which is contained in eqs. (154), (160) and (161) may be evaluated almost analytically on the basis of eqs. (42)$\sim$(44) or eqs. (63) and (65).

5. Maximum value of the energy spectral density of the modified quasi-stationary random process

Since the power spectral density $S_{H\xi}(\tau\,;\,\omega, h, \tau_d)$ of the modified quasi-stationary random process is not positive in the open time domain (τ_d, ∞), the time τ_{max} at which the energy spectral density $S_{E\xi}(\tau\,;\,\omega, h, \tau_d)$ of the modified quasi-stationary random process takes the maximum value exists in the right-closed interval $(0, \tau_d]$, namely

$$0<\tau_{max}\leqq\tau_d \tag{165}$$

If τ_{max} exists in the open interval $(0, \tau_d)$ it may be a stationary point of $S_{E\xi}(\tau; \omega, h, \tau_d)$, that is, a zero of $S_{H\xi}(\tau; \omega, h, \tau_d)$ such that

$$\begin{aligned} S_{E\xi\tau}^{(1)}(\tau_{max}; \omega, h, \tau_d) &= S_{H\xi}(\tau_{max}; \omega, h, \tau_d) = 0 \\ S_{E\xi\tau}^{(2)}(\tau_{max}; \omega, h, \tau_d) &= S_{H\xi\tau}^{(1)}(\tau_{max}; \omega, h, \tau_d) < 0 \end{aligned} \tag{166}$$

because both $S_{E\xi}(\tau; \omega, h, \tau_d)$ and $S_{H\xi}(\tau; \omega, h, \tau_d)$ in the open domain $(0, \tau_d)$ may be the continuous, differentiable functions of time as shown in eqs. (58) and (59). However, if τ_{max} is equal to τ_d the power spectral density $S_{H\xi}(\tau; \omega, h, \tau_d)$ is not zero at τ_{max} since it is generally discontinuous at τ_d as shown in eqs. (50) and (51).

For convenience in evaluating the maximum value of the energy spectral density $S_{E\xi}(\tau_{max}; \omega, h, \tau_d)$ the following two cases are considered separately; the first is the case where the envelope of the original quasi-stationary random process is time-invariant, namely $a(\tau)=1$ and the second is the case where the envelope is time-variant, that is, $a(\tau)\not\equiv 1$. Thus in the first case the quasi-stationary random excitations are given by the form $D(\tau; R^1{}_{0\tau_d})\psi(\tau)$ while in the second case they are expressed in the form $D(\tau; R^1{}_{0\tau_d})a(\tau)\psi(\tau)$.

5.1 *The case of time-invariant envelope, $a(\tau)=1$*

In this case the Fourier transform of the envelope $a(\tau)$ of the original quasi-stationary excitations is given by

$$A(j\omega) = 2\pi\delta(\omega) \subset a(\tau) = 1 \tag{167}$$

where $\delta(\omega)$ denotes Dirac's delta-function. Substituting eq. (167) in eq. (44), the complex-valued function $c(\mu, \tau; \omega, h, \tau_d)$ and its absolute square are obtained as follows:

$$c(\mu, \tau; \omega, h, \tau_d) = \exp(-h|\omega|\tau)\frac{\exp\{(h|\omega| + j(\mu-\omega))\tau_m\} - 1}{h|\omega| + j(\mu-\omega)} \tag{168}$$

$$\begin{aligned} &|c(\mu, \tau; \omega, h, \tau_d)|^2 \\ &= \exp(-2h|\omega|\tau)\frac{\exp(2h|\omega|\tau_m) - 2\exp(h|\omega|\tau_m)\cos(\mu-\omega)\tau_m + 1}{(h\omega)^2 + (\mu-\omega)^2} \end{aligned} \tag{169}$$

Hence from eq. (42) the energy spectral density in the time domain $[0, \tau_d]$ is expressed as follows by considering the relation $\tau_m = \tau$ which is valid in this time domain:

$$\begin{aligned} S_{E\xi}(\tau; \omega, h, \tau_d) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\mu S_\psi(\mu) \\ &\cdot \frac{1 + \exp(-2h|\omega|\tau) - \exp(-h|\omega|\tau)\{\exp(j(\omega-\mu)\tau) + \exp(-j(\omega-\mu)\tau)\}}{(h\omega)^2 + (\omega-\mu)^2} \\ &\qquad 0 \leqq \tau \leqq \tau_d \end{aligned} \tag{170}$$

On the other hand, by substituting $a(\tau)=1$ in eq. (53) the functions $Rb(\mu, \tau; \omega, h)$ and $|b(\mu, \tau; \omega, h)|^2$ are determined respectively as follows:

$$\begin{aligned} Rb(\mu, \tau; \omega, h) &= \frac{1}{2}\int_0^\tau \exp(-h|\omega|\kappa)\{\exp(j(\omega-\mu)\kappa) + \exp(-j(\omega-\mu)\kappa)\}d\kappa \\ &= \frac{h|\omega|\{1 - \exp(-h|\omega|\tau)\cos(\omega-\mu)\tau\} + (\omega-\mu)\exp(-h|\omega|\tau)\sin(\omega-\mu)\tau}{(h\omega)^2 + (\omega-\mu)^2} \end{aligned} \tag{171}$$

$$|b(\mu,\tau\,;\,\omega,h)|^2=\int_0^\tau d\nu_2\int_{-\nu_2}^{\tau-\nu_2}d\nu_1\exp(-h|\omega|(\nu_1+2\nu_2))\exp(-j(\omega-\mu)\nu_1)$$
$$=\frac{1-\exp\{-(h|\omega|-j(\omega-\mu))\tau\}}{h|\omega|-j(\omega-\mu)}\int_0^\tau\exp\{-(h|\omega|+j(\omega-\mu))\nu_2\}d\nu_2$$
$$=\frac{1+\exp(-2h|\omega|\tau)-2\exp(-h|\omega|\tau)\cos(\omega-\mu)\tau}{(h\omega)^2+(\omega-\mu)^2} \tag{172}$$

Substituting eqs. (171) and (172) in eq. (50) the power spectral density in the time domain $[0,\tau_d)$ is expressed as follows:

$$S_{H\xi}(\tau\,;\,\omega,h,\tau_d)=\frac{1}{\pi}\int_{-\infty}^{\infty}d\mu S_\phi(\mu)\frac{1}{(h\omega)^2+(\omega-\mu)^2}$$
$$\cdot\{h|\omega|(\exp(-h|\omega|\tau)\cos(\omega-\mu)\tau-\exp(-2h|\omega|\tau))$$
$$+(\omega-\mu)\exp(-h|\omega|\tau)\sin(\omega-\mu)\tau\},\qquad 0\leqq\tau<\tau_d \tag{173-a}$$

On the other hand, the spectral density in the time domain (τ_d,∞) is expressed as follows by making use of eqs. (51) and (172):

$$S_{H\xi}(\tau\,;\,\omega,h,\tau_d)=-\frac{h|\omega|\exp(-2h|\omega|(\tau-\tau_d))}{\pi}\int_{-\infty}^{\infty}d\mu S_\phi(\mu)$$
$$\cdot\frac{1+\exp(-2h|\omega|\tau_d)-2\exp(-h|\omega|\tau_d)\cos(\omega-\mu)\tau_d}{(h\omega)^2+(\omega-\mu)^2}\leqq 0,\qquad \tau_d<\tau \tag{173-b}$$

It is easily seen that the values of the two functions given by eqs. (173-a) and (173-b) do not coincide with each other at the boundary point $\tau=\tau_d$.

In the special case where the power spectral density $S_\phi(\omega)$ of the stationary random process is white, that is,

$$S_\phi(\omega)=c^2 \tag{174}$$

the energy spectral density of the modified quasi-stationary process given by eq. (170) can be reduced to the following form;

$$S_{E\xi}(\tau\,;\,\omega,h,\tau_d)=c^2(I_1-I_2-I_3) \tag{175}$$

and

$$I_1=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1+\exp(-2h|\omega|\tau)}{(h\omega)^2+(\omega-\mu)^2}d\mu \tag{176}$$

$$I_2=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\exp\{-(h|\omega|-j(\omega-\mu))\tau\}}{(h\omega)^2+(\omega-\mu)^2}d\mu \tag{177}$$

$$I_3=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\exp\{-(h|\omega|+j(\omega-\mu))\tau\}}{(h\omega)^2+(\omega-\mu)^2},\qquad \tau\geqq 0,\quad h\geqq 0 \tag{178}$$

in which two zeros of the common denominator of the integrands are given by

$$\lambda_1=\omega+jh|\omega|,\qquad \lambda_2=\lambda_1{}^*=\omega-jh|\omega| \tag{179}$$

Supposing the variable μ to be a complex number, the integrand in eq. (176) is analytic in the full complex plane except for two poles λ_1 and λ_2, and the integrands in eqs. (177) and (178) are analytic except for a relevant pole λ_2 or λ_1 in the lower and upper half-plane respectively, provided that τ and h are positive. When $|\mu|$ tends to infinity, the order of each integrand in eqs. (176) $\sim$(178) is $0(\mu^{-2})$ in the relevant complex plane. Hence by applying the residue theorem the definite integrals given by eqs. (176)~(178) can be easily evaluated as follows:

$$I_1=\frac{1+\exp(-2h|\omega|\tau)}{2h|\omega|}, \qquad I_2=I_3=\frac{\exp(-2h|\omega|\tau)}{2h|\omega|}, \qquad h>0 \tag{180}$$

Substituting eq. (180) in eq. (175) the energy spectral density in the time domain $[0, \tau_d]$ is given by

$$S_{E\xi}(\tau\,;\,\omega,h,\tau_d)=\frac{c^2(1-\exp(-2h|\omega|\tau))}{2h|\omega|}, \qquad 0\leqq\tau\leqq\tau_d \tag{181}$$

Hence the maximum τ_{max} is clearly τ_d and the maximum value of the energy spectral density is obtained as follows:

$$\sup_{T} S_{E\xi}(\tau\,;\,\omega,h,\tau_d)=\frac{c^2(1-\exp(-2h|\omega|\tau_d))}{2h|\omega|}, \qquad h>0 \tag{182}$$

In particular, for the case where the damping parameter $h=0$, the maximum value of the energy spectral density is obtained by taking the limit $h\to 0$ in eq. (182) as follows:

$$\sup_{T} S_{E\xi}(\tau\,;\,\omega,0,\tau_d)=\tau_d c^2 \tag{183}$$

In another special case where the damping parameter $h=0$ the energy and power spectral densities given by eqs. (42), (173-a) and (173-b) are expressed in the following forms respectively:

$$S_{E\xi}(\tau\,;\,\omega,0,\tau_d)=\frac{1}{\pi}\int_{-\infty}^{\infty}S_\psi(\mu)\frac{1-\cos(\omega-\mu)\tau_m}{(\omega-\mu)^2}d\mu=\frac{1}{\pi}S_\psi(\omega)*\frac{1-\cos\omega\tau_m}{\omega^2}, \qquad 0\leqq\tau \tag{184}$$

$$S_{H\xi}(\tau\,;\,\omega,0,\tau_d)=\frac{1}{\pi}\int_{-\infty}^{\infty}S_\psi(\mu)\frac{\sin(\omega-\mu)\tau}{(\omega-\mu)}d\mu=\frac{1}{\pi}S_\psi(\omega)*\frac{\sin\omega\tau}{\omega}, \qquad 0\leqq\tau<\tau_d \tag{185-a}$$

$$S_{H\xi}(\tau\,;\,\omega,0,\tau_d)=0, \qquad \tau_d<\tau \tag{185-b}$$

From eq. (185-a) it is found that for an arbitrary power spectral density $S_\psi(\omega)$ the power spectral density $S_{H\xi}(\tau\,;\,\omega,h,\tau_d)$ of the modified quasi-stationary random process is not always positive and hence the energy spectral density $S_{E\xi}(\tau\,;\,\omega,h,\tau_d)$ does not always increase in the time domain $[0, \tau_d]$, even in the case where the envelope $a(\tau)$ is time-invariant and the damping parameter h is zero. This phenomenon may occur in the case where the power spectral density $S_\psi(\omega)$ is sharp, τ is comparatively small and a certain relation holds between the frequency parameter ω, the predominant frequency ω_p of $S_\psi(\omega)$ and time τ. From eq. (51) it is generally shown that the power spectral density $S_{H\xi}(\tau\,;\,\omega,h,\tau_d)$ in the domain (τ_d, ∞) is identically zero, hence the energy spectral density $S_{E\xi}(\tau\,;\,\omega,h,\tau_d)$ is constant in this time domain as far as the damping parameter h is zero.

If $\tau_m=\min(\tau, \tau_d)$ tends to infinity in eqs. (184) and (185-a) the following relations are obtained;

$$S_{E\xi}(\tau,\omega,0,\tau_d)\to\tau_m S_\psi(\omega), \qquad \tau_m\to\infty \tag{186-a}$$

$$S_{H\xi}(\tau, \omega, 0, \tau_d) \rightarrow S_\psi(\omega) \geqq 0, \qquad \tau \rightarrow \infty, \quad 0 \leqq \tau < \tau_d \tag{186-b}$$

by making use of eq. (120) and the limiting formula,

$$\lim_{\tau_m \rightarrow \infty} \frac{1}{\pi} \frac{\sin \omega \tau_m}{\omega} = \delta(\omega)$$

Hence the maximum value of the energy spectral density is expressed in the following form in the case of zero damping and sufficiently large τ_d.

$$\sup_T S_{E\xi}(\tau, \omega, 0, \tau_d) \rightarrow \tau_d S_\psi(\omega), \qquad \tau_d \rightarrow \infty \tag{187}$$

In the case where $1 \gg h \geqq 0$ and $S_\psi(\omega)$ is a sufficiently flat spectrum in the wide frequency range, the maximum value of energy spectral density may be approximately evaluated by the following procedure: by substituting $a(\tau) = 1$ and the approximate expression of the auto-correlation function $R_\psi(\lambda) = S_\psi(\omega)\delta(\lambda)$ in eq. (55) the power spectral density in the domain $[0, \tau_d)$ is approximately expressed as

$$\begin{aligned} S_{H\xi}(\tau; \omega, h, \tau_d) &= 2S_\psi(\omega)\left(\frac{1}{2} - h|\omega|\int_0^\tau \exp(-2h|\omega|\kappa)\,d\kappa\right) \\ &= S_\psi(\omega)\exp(-2h|\omega|\tau) \geqq 0 \end{aligned} \tag{188}$$

Hence the maximum value of the energy spectral density is approximately given by the value at τ_d, namely

$$\sup_T S_{E\xi}(\tau; \omega, h, \tau_d) \fallingdotseq S_{E\xi}(\tau_d; \omega, h, \tau_d) \tag{189}$$

On the other hand, by considering the condition $1 \gg h > 0$ and by making use of eq. (181) the energy spectral density in the time domain $[0, \tau_d]$ can be approximately evaluated by the following equation;[22]

$$\begin{aligned} S_{E\xi}(\tau; \omega, h, \tau_d) &\fallingdotseq \frac{S_\psi(\omega)}{2\pi}\int_{-\infty}^{\infty} d\mu \\ &\cdot \frac{1 + \exp(-2h|\omega|\tau) - \exp(-h|\omega|\tau)\{\exp(j(\omega-\mu)\tau) + \exp(-j(\omega-\mu)\tau)\}}{(h\omega)^2 + (\omega-\mu)^2} \\ &= \frac{1 - \exp(-2h|\omega|\tau)}{2h|\omega|} S_\psi(\omega), \qquad 0 < h \ll 1, \quad 0 \leqq \tau \leqq \tau_d \end{aligned} \tag{190}$$

In particular, for the case where $h = 0$ the above equation reduces to the following form:

$$S_{E\xi}(\tau; \omega, 0, \tau_d) = \tau S_\psi(\omega) \tag{191}$$

Eqs. (190) and (191) are also obtained by integrating eq. (188) with the zero initial condition.

From eqs. (190) and (191), the maximum value of the energy spectral density is approximately given by the following formulae for the case of the flat power spectral density $S_\psi(\omega)$ and zero or slight damping parameter.

$$\sup_T S_{E\xi}(\tau; \omega, h, \tau_d) = \frac{1 - \exp(-2h|\omega|\tau_d)}{2h|\omega|} S_\psi(\omega), \qquad 0 < h \leqq 1 \tag{192}$$

$$\sup_T S_{E\xi}(\tau; \omega, 0, \tau_d) = \tau_d S_\psi(\omega), \qquad h = 0 \tag{193}$$

Finally, in the case where $a(\tau)=1$ and $S_\phi(\omega)$ is expressed as a rational function given by

$$S_\phi(\omega)=c^2\frac{\prod_i\left[1+\left(\frac{\omega}{\omega_i}\right)^{2\beta_i}\right]}{\prod_i\left[1+\left(\frac{\omega}{\omega_i}\right)^{2\alpha_i}\right]},\qquad \sum_i\alpha_i\geqq\sum_i\beta_i \tag{194}$$

the analytical expressions of the energy and spectral densities of the modified quasi-stationary random excitations and the iterative method to estimate the maximum value of the energy spectral density are considered.

The zeros of the algebraic equation,

$$1+\lambda^{2\gamma}=0 \tag{195}$$

are given by

$$\exp\left\{\frac{j\pi(1+2\nu)}{2\gamma}\right\},\qquad \nu=0,1,\cdots\cdots,2\gamma-1 \tag{196}$$

and these zeros consist of the following two sets of complex conjugate numbers;

$$(\lambda^1{}_\gamma,\lambda^2{}_\gamma,\cdots\cdots,\lambda^\gamma{}_\gamma),\qquad (\lambda^1{}_\gamma{}^*,\lambda^2{}_\gamma{}^*,\cdots\cdots,\lambda^\gamma{}_\gamma{}^*) \tag{197}$$

in which the first and the second sets are supposed to be in the upper and the lower half-plane, respectively[23].

Denoting

$$\mu^\kappa{}_{\alpha_i}=\omega_i\lambda^\kappa{}_{\alpha_i},\qquad \kappa=1,2,\cdots\cdots,\alpha_i,\qquad i=1,2,\cdots\cdots \tag{198}$$

$$\mu^\nu{}_{\beta_i}=\omega_i\lambda^\nu{}_{\beta_i},\qquad \nu=1,2,\cdots\cdots,\beta_i,\qquad i=1,2,\cdots\cdots \tag{199}$$

the power spectral density $S_\phi(\omega)$ given by eq. (194) is written as follows:

$$S_\phi(\omega)=\psi_\phi(\omega)\psi_\phi{}^*(\omega) \tag{200}$$

$$\psi_\phi(\omega)=c\prod_i\omega_i{}^{(\alpha_i-\beta_i)}\frac{\prod_{\nu=1}^{\beta_i}(\omega-\mu^\nu{}_{\beta_i})}{\prod_{\kappa=1}^{\alpha_i}(\omega-\mu^\kappa{}_{\alpha_i})} \tag{201}$$

where $\psi_\phi(\omega)$ and $\psi_\phi{}^*(\omega)$ are analytic, bounded and non-zero in the lower and the upper half-plane, respectively.

Here, in obtaining the definite expression of the energy spectral density in the time domain $[0,\tau_d]$, eq. (170) is rewritten in temrs of the following three integrals:

$$S_{E\xi}(\tau;\omega,h,\tau_d)=(1+\exp(-2h|\omega|\tau))J_1-\exp(-h|\omega|\tau)(J_2+J_3) \tag{202}$$

$$J_1=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{S_\phi(\mu)}{(h\omega)^2+(\omega-\mu)^2}d\mu \tag{203}$$

$$J_2=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\exp(j(\omega-\mu)\tau)S_\phi(\mu)}{(h\omega)^2+(\omega-\mu)^2}d\mu \tag{204}$$

$$J_3=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\exp(-j(\omega-\mu)\tau)S_\phi(\mu)}{(h\omega)^2+(\omega-\mu)^2}d\mu \tag{205}$$

Since when $|\mu|$ tends to infinity each integrand in eqs. (203)~(205) is of the order of $0(\mu^{-2})$ in the full plane, lower half-plane and upper half-plane, respectively, and all the singular points of these integrands are poles, the integrals given by eqs. (203)~(205) can be evaluated by making use of the residue theorem. Here, for the sake of simplicity, it is assumed that all $\mu^{\kappa}\alpha_i$'s are simple roots and $\omega+jh|\omega|$ does not coincide with any one of $\mu^{\kappa}\alpha_i$'s. Hence the integrals in eqs. (202)~(205) are obtained as follows:

$$\begin{aligned} J_1 &= \frac{S_\phi(\omega+jh|\omega|)}{2h|\omega|} + j\sum_{i,\kappa}\frac{R(S_\phi(\mu^\kappa\alpha_i))}{(h\omega)^2+(\omega-\mu^\kappa\alpha_i)^2} \\ &= \frac{S_\phi(\omega-jh|\omega|)}{2h|\omega|} - j\sum_{i,\kappa}\frac{R(S_\phi(\mu^\kappa\alpha_i^*))}{(h\omega)^2+(\omega-\mu^\kappa\alpha_i^*)^2} \\ &= \frac{RS_\phi(\omega+jh|\omega|)}{2h|\omega|} - \sum_{i,\kappa} I\frac{R(S_\phi(\mu^\kappa\alpha_i))}{(h\omega)^2+(\omega-\mu^\kappa\alpha_i)^2} \end{aligned} \tag{206}$$

$$J_2=J_3^*=\frac{\exp(-h|\omega|\tau)S_\phi(\omega-jh|\omega|)}{2h|\omega|} - j\sum_{i,\kappa}\frac{R(S_\phi(\mu^\kappa\alpha_i^*))\exp(j(\omega-\mu^\kappa\alpha_i^*)\tau)}{(h\omega)^2+(\omega-\mu^\kappa\alpha_i^*)^2} \tag{207}$$

and

$$J_2+J_3=\frac{\exp(-h|\omega|\tau)RS_\phi(\omega+jh|\omega|)}{h|\omega|} - 2\sum_{i,\kappa} I\frac{R(S_\phi(\mu^\kappa\alpha_i))\exp(-j(\omega-\mu^\kappa\alpha_i)\tau)}{(h\omega)^2+(\omega-\mu^\kappa\alpha_i)^2} \tag{208}$$

in which $R(S_\phi(\mu^\kappa\alpha_i))$ denotes the residue of $S_\phi(\omega)$ at the pole $\mu^\kappa\alpha_i$. With the above-mentioned assumptions the residue of eq. (194) at the pole $\mu^q\alpha_p$ is expressed as follows:

$$\begin{aligned} R(S_\phi(\mu^q\alpha_p)) &= \frac{c^2\prod_i\left[1+\left(\frac{\mu^q\alpha_p}{\omega_i}\right)^{2\beta_i}\right]}{\frac{2\alpha_p}{\omega_p}\left(\frac{\mu^q\alpha_p}{\omega_p}\right)^{2\alpha_p-1}\prod_{i\neq p}\left[1+\left(\frac{\mu^q\alpha_p}{\omega_i}\right)^{2\alpha_i}\right]} \\ &= \frac{c^2\omega_p{}^{2(\alpha_p-\beta_p)}\prod_{\nu=1}^{\beta_p}|\mu^q\alpha_p-\mu^\nu\beta_p|^2}{2jI\mu^q\alpha_p\prod_{\kappa\neq q}^{\alpha_p}|\mu^q\alpha_p-\mu^\kappa\alpha_p|^2}\prod_{i\neq p}\omega_i{}^{2(\alpha_i-\beta_i)}\frac{\prod_{\nu=1}^{\beta_i}|\mu^q\alpha_p-\mu^\nu\beta_i|^2}{\prod_{\kappa=1}^{\alpha_i}|\mu^q\alpha_p-\mu^\kappa\alpha_i|^2} \end{aligned} \tag{209}$$

By making use of eqs. (206)~(208) the energy spectral density in the time domain $[0, \tau_d]$ is given by the following equation:

$$\begin{aligned} S_{E\xi}(\tau;\omega,h,\tau_d) &= \frac{1-\exp(-2h|\omega|\tau)}{2h|\omega|}RS_\phi(\omega+jh|\omega|) \\ &\quad -(1+\exp(-2h|\omega|\tau))\sum_{i,\kappa} I\frac{R(S_\phi(\mu^\kappa\alpha_i))}{(h\omega)^2+(\omega-\mu^\kappa\alpha_i)^2} \\ &\quad +2\exp(-h|\omega|\tau)\sum_{i,\kappa} I\frac{R(S_\phi(\mu^\kappa\alpha_i))\exp(-j(\omega-\mu^\kappa\alpha_i)\tau)}{(h\omega)^2+(\omega-\mu^\kappa\alpha_i)^2}, \\ &\qquad 0\leqq\tau\leqq\tau_d \end{aligned} \tag{210}$$

Similarly the power spectral density in the time domain $[0, \tau_d)$ given by eq. (173-a) is expressed by

$$\begin{aligned} S_{H\xi}(\tau;\omega,h,\tau_d) &= -2h|\omega|\exp(-2h|\omega|\tau)J_1+h|\omega|\exp(-h|\omega|\tau)(J_2+J_3) \\ &\quad +\exp(-h|\omega|\tau)(J_4+J_5) \end{aligned} \tag{211}$$

in which J_1, J_2 and J_3 are defined by eqs. (203)~(205) respectively and, J_4 and J_5 are defined by the following equations:

$$J_4=\frac{1}{2\pi j}\int_{-\infty}^{\infty}\frac{(\omega-\mu)\exp(j(\omega-\mu)\tau)}{(h\omega)^2+(\omega-\mu)^2}S_\phi(\mu)d\mu \tag{212}$$

$$J_5=\frac{-1}{2\pi j}\int_{-\infty}^{\infty}\frac{(\omega-\mu)\exp(-j(\omega-\mu)\tau)}{(h\omega)^2+(\omega-\mu)^2}S_\phi(\mu)d\mu \tag{213}$$

The above two integrals can be evaluated by applying the residue theorem as in eqs. (203)~(205) as follows:

$$J_4=J_5{}^*=\frac{\exp(-h|\omega|\tau)}{2}S_\phi(\omega-jh|\omega|)$$
$$-\sum_{i,\kappa}\frac{R(S_\phi(\mu^\kappa\alpha_i{}^*))(\omega-\mu^\kappa\alpha_i{}^*)\exp(j(\omega-\mu^\kappa\alpha_i{}^*)\tau)}{(h\omega)^2+(\omega-\mu^\kappa\alpha_i{}^*)^2} \tag{214}$$

Substituting eqs. (206), (208) and (214) in eq. (211) the power spectral density in the time domain $[0, \tau_d)$ is written in the following form:

$$S_{H\xi}(\tau\,;\,\omega,h,\tau_d)=\exp(-2h|\omega|\tau)\boldsymbol{R}S_\phi(\omega+jh|\omega|)$$
$$-2\exp(-h|\omega|\tau)\sum_{i,\kappa}\boldsymbol{R}\frac{R(S_\phi(\mu^\kappa\alpha_i))(\omega-\mu^\kappa\alpha_i)\exp(-j(\omega-\mu^\kappa\alpha_i)\tau)}{(h\omega)^2+(\omega-\mu^\kappa\alpha_i)^2}$$
$$+2h|\omega|\exp(-h|\omega|\tau)$$
$$\cdot\sum_{i,\kappa}\boldsymbol{I}\frac{R(S_\phi(\mu^\kappa\alpha_i))\{\exp(-h|\omega|\tau)-\exp(-j(\omega-\mu^\kappa\alpha_i)\tau)\}}{(h\omega)^2+(\omega-\mu^\kappa\alpha_i)^2},\qquad 0\leqq\tau<\tau_d \tag{215}$$

Particularly for the case where $S_\phi(\omega)=c^2$ eqs. (210) and (215) reduce to eq. (181) and its time derivative respectively.

As another special case, substituting the damping parameter $h=0$ in eqs. (210) and (215) the following equations are obtained:

$$S_{E\xi}(\tau\,;\,\omega,0,\tau_d)=\tau S_\phi(\omega)-2\sum_{i,\kappa}\boldsymbol{I}\frac{R(S_\phi(\mu^\kappa\alpha_i))}{(\omega-\mu^\kappa\alpha_i)^2}$$
$$+2\sum_{i,\kappa}\boldsymbol{I}\frac{R(S_\phi(\mu^\kappa\alpha_i))\exp(-j(\omega-\mu^\kappa\alpha_i)\tau)}{(\omega-\mu^\kappa\alpha_i)^2},\qquad 0\leqq\tau\leqq\tau_d \tag{216}$$

$$S_{H\xi}(\tau\,;\,\omega,0,\tau_d)=S_\phi(\omega)-2\sum_{i,\kappa}\boldsymbol{R}\frac{R(S_\phi(\mu^\kappa\alpha_i))\exp(-j(\omega-\mu^\kappa\alpha_i)\tau)}{(\omega-\mu^\kappa\alpha_i)},\qquad 0\leqq\tau<\tau_d \tag{217}$$

It is easily verified by simple manipulation that eqs. (216) and (217) agree to eqs. (184) and (185-a), respectively.

In the above, the analytical expressions of the energy and power spectral densities associated with the time domain in which the maximum of the energy spectral density is contained are obtained. However it is not easy to analytically determine the maximum and the maximum value of the energy spectral density for the general case where $1>h\geqq 0$ and $S_\phi(\omega)$ is an arbitrary rational function expressed as in eq. (194). Hence numerical estimation of the energy spectral density given by eq. (210) in the time domain $[0, \tau_d]$ seems to be generally necessary to find its maximum value.

In the case of the time-invariant envelope $a(\tau)=1$, however, if $1>h\geqq 0$ and $S_\phi(\omega)$ is a sufficiently flat spectrum over the wide frequency range the random responses defined by eq. (24) seem to belong to the divergent process. Therefore the maximum τ_{max} of the energy spectral density may be approximately estimated as the end point τ_d of the duration time of such quasi-stationary random excitations. Then in an approximate sense, the maximum value of the energy spectral density is given by the following equation:

$$\begin{aligned}\sup_{\tau} S_{E\xi}(\tau\,;\,\omega,h,\tau_d) &\doteqdot S_{E\xi}(\tau_d\,;\,\omega,h,\tau_d)\\ &=\frac{1-\exp(-2h|\omega|\tau_d)}{2h|\omega|}RS_\psi(\omega+jh|\omega|)-(1+\exp(-2h|\omega|\tau_d))\sum_{i,\kappa}I\frac{R(S_\psi(\mu^\kappa\alpha_i))}{(h\omega)^2+(\omega-\mu^\kappa\alpha_i)^2}\\ &+2\exp(-h|\omega|\tau_d)\sum_{i,\kappa}I\frac{R(S_\psi(\mu^\kappa\alpha_i))\exp(-j(\omega-\mu^\kappa\alpha_i)\tau)}{(h\omega)^2+(\omega-\mu^\kappa\alpha_i)^2}\end{aligned} \tag{218}$$

Of course, it is possible to determine the more accurate value of τ_{max} by means of the successive approximation procedure starting from the first approximate value $\tau^{(1)}=\tau_d$. If τ_{max} is equal to τ_d the power spectral density should not be negative at τ_d. Hence if $S_{H\xi}(\tau_d\,;\,\omega,h,\tau_d)$ is negative τ_{max} must exist in the open time domain $(0,\tau_d)$ and eq. (166) should be valid, namely

$$S_{H\xi}(\tau_{max}\,;\,\omega,h,\tau_d)=0,\qquad 0<\tau_{max}<\tau_d \tag{219}$$

Substituting $a(\tau)=1$ in eq. (55) the above equation is rewritten as follows:

$$\begin{aligned}&\int_0^{\tau_{max}}\exp(-h|\omega|\kappa)R_\psi(\kappa)\cos\omega\kappa d\kappa\\ &=h|\omega|\int_0^{\tau_{max}}d\kappa_1\int_0^{\tau_{max}}d\kappa_2\exp(-h|\omega|(\kappa_1+\kappa_2))R_\psi(\kappa_1-\kappa_2)\exp(-j\omega(\kappa_1-\kappa_2))\\ &=h|\omega|\int_0^{\tau_{max}}d\nu_2\exp(-2h|\omega|\nu_2)\int_{-\nu_2}^{\tau_{max}-\nu_2}d\nu_1\exp(-h|\omega|\nu_1)R_\psi(\nu_1)\cos\omega\nu_1\end{aligned} \tag{220}$$

On the basis of eq. (220), the following formulae for a successive approximation procedure of the maximum τ_{max} may be obtained:

$$\tau^{(n+1)}=\tau^{(n)}+\Delta\tau^{(n)},\qquad \tau^{(1)}=\tau_d,\qquad n=1,2,\cdots\cdots \tag{221}$$

$$\Delta\tau^{(n)}=\frac{h|\omega|\int_0^{\tau^{(n)}}d\nu_2\exp(-2h|\omega|\nu_2)\int_{-\nu_2}^{\tau^{(n)}-\nu_2}d\nu_1\exp(-h|\omega|\nu_1)R_\psi(\nu_1)\cos\omega\nu_1-\int_0^{\tau^{(n)}}d\kappa\exp(-h|\omega|\kappa)R_\psi(\kappa)\cos\omega\kappa}{\exp(-h|\omega|\tau^{(n)})R_\psi(\tau^{(n)})\cos\omega\tau^{(n)}+2h|\omega|\left(h|\omega|\int_0^{\tau^{(n)}}d\nu_2\exp(-2h|\omega|\nu_2)\cdot\int_{-\nu_2}^{\tau^{(n)}-\nu_2}d\nu_1\exp(-h|\omega|\nu_1)R_\psi(\nu_1)\cos\omega\nu_1-\int_0^{\tau^{(n)}}d\kappa\exp(-h|\omega|\kappa)R_\psi(\kappa)\cos\omega\kappa\right)} \tag{222}$$

in which the auto-correlation function $R_\psi(\tau)$ is given by the following formula:

$$\begin{aligned}R_\psi(\tau)&=\frac{1}{2\pi}\int_{-\infty}^{\infty}S_\psi(\omega)\exp(j\omega\tau)d\omega\\ &=j\sum_{i,\kappa}R(S_\psi(\mu^\kappa\alpha_i))\exp(j\mu^\kappa\alpha_i|\tau|)\\ &=-\sum_{i,\kappa}(RR(S_\psi(\mu^\kappa\alpha_i))\exp(-I\mu^\kappa\alpha_i|\tau|)\sin(R\mu^\kappa\alpha_i|\tau|)\\ &\quad+IR(S_\psi(\mu^\kappa\alpha_i))\exp(-I\mu^\kappa\alpha_i|\tau|)\cos(R\mu^\kappa\alpha_i|\tau|))\end{aligned} \tag{223}$$

5.2. The case of time-variant envelope, $a(\tau)\not\equiv 1$

In general, for an arbitrary envelope $a(\tau)$ of the original quasi-stationary random excitations it is difficult to analytically obtain the maximum value of the energy spectral density of the modified quasi-stationary random process. Therefore the maximum value of the energy spectral density must be found based upon the numerical evaluation of the energy spectral density in the time domain $[0, \tau_d]$. Here it is noted that the maximum τ_{max} of the energy spectral density of the modified quasi-stationary random process always exists in the time domain $[0, \tau_d]$. If the approximate value of τ_{max} is known, the more exact value of τ_{max} may be obtained by means of the successive approximation procedure as in the preceding subsection.

In connexion with the above-mentioned procedure, the analytical expressions of the energy and power spectral densities which are given respectively by eqs. (63) and (64) together with eq. (65) may be available because the convolution integrals associated with the finite time domain which are contained in eqs. (63), (64) and (65) may be suitable for the numerical evaluations. Then if the analytical expression of the auto-correlation function $R_\psi(\tau)$ is given, both the energy and power spectral densities can be expressed in the formula containing the double convolution integral according to eqs. (63)~(65).

In the following, the power spectral density $S_\psi(\omega)$ of the stationary random process is supposed to be expressed as the product of the rational function given by eq. (194) and the band-limiting operator defined by

$$\begin{aligned} &B(\omega\,;\,\omega_c,\,\omega_b)=D(\omega\,;\,R^1{}_{\omega_L\omega_U})+D(\omega\,;\,R^1{}_{-\omega_U-\omega_L}) \\ &\omega_L=\omega_c-\omega_b, \qquad \omega_U=\omega_c+\omega_b, \qquad \omega_c>\omega_b>0 \end{aligned} \tag{224}$$

where the operators appearing in the right-hand side are the cutoff operators associated with the frequency variable, defined as in eq. (3), and ω_c, ω_b denote the center frequency and half band-width of the cutoff operator, respectively. Then, $S_\psi(\omega)$ is expressed as

$$S_\psi(\omega)=B(\omega\,;\,\omega_c,\,\omega_b)S_\psi'(\omega) \tag{225}$$

$$S_\psi'(\omega)=\psi_\psi(\omega)\psi_\psi^*(\omega) \tag{226}$$

in which $\psi_\psi(\omega)$ is the complex-valued function defined by eq. (201).

In evaluating $R_\psi(\tau)$ in eq. (65) the following two cases are considered according to the order of $\psi_\psi(\omega)$ or $S_\psi'(\omega)$ when $|\omega|$ tends to infinity; namely, supposing δ to be a non-negative integer of the order of $\psi_\psi(\omega)$ and $S_\psi'(\omega)$ as $|\omega|$ tending to infinity, expressed as

$$\begin{aligned} &0(\psi_\psi(\omega))=0(\omega^{-\delta}) \qquad |\omega|\to\infty \\ &0(S_\psi'(\omega))=0(\omega^{-2\delta}) \qquad |\omega|\to\infty, \qquad \delta\geqq 0 \end{aligned} \tag{227}$$

then, the first is the case where δ is a positive integer and the second is the case where δ is zero. As in the preceding subsection it is assumed that all the poles of $S_\psi'(\omega)$ are simple and that they consist of the two sets of complex numbers $\{\mu^\kappa{}_{\alpha_i}\}$ and $\{\mu^\kappa{}_{\alpha_i}{}^*\}$ given by eqs. (196)~(198). According to the above-mentioned classification, the function $f(\tau\,;\,\omega,\,h)$ defined by eq. (62) or eq. (65) is expressed in the definite form available for the numerical evaluation.

Case A, $\delta\geqq 1$

In this case the auto-correlation function $R_\psi(\tau)$ which is contained in the integral representation of $f(\tau\,;\,\omega, h)$ as in eq. (65) is expressed in the following form by making use of the residue theorem and the convolution theorem of the Fourier transform :

$$\begin{aligned} R_\psi(\tau) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_\psi(\mu)\exp(j\mu\tau)d\mu = \frac{1}{2\pi}\int_{-\infty}^{\infty} B(\mu\,;\,\omega_c, \omega_b)S_\psi{}'(\mu)\exp(j\mu\tau)d\mu \\ &= j\sum_{i,\kappa} RS_\psi{}'(\mu^\kappa\alpha_i)\exp(j\mu^\kappa\alpha_i|\tau|)*\frac{2}{\pi}\frac{\sin\omega_b\tau\cos\omega_c\tau}{\tau} \\ &= -\sum_{i,\kappa} I(RS_\psi{}'(\mu^\kappa\alpha_i)\exp(j\mu^\kappa\alpha_i|\tau|))*\frac{2}{\pi}\frac{\sin\omega_b\tau\cos\omega_c\tau}{\tau} \\ &= -\sum_{i,\kappa}(RRS_\psi{}'(\mu^\kappa\alpha_i)\exp(-I\mu^\kappa\alpha_i|\tau|)\sin(R\mu^\kappa\alpha_i|\tau|) \\ &\qquad + IRS_\psi{}'(\mu^\kappa\alpha_i)\exp(-I\mu^\kappa\alpha_i|\tau|)\cos(R\mu^\kappa\alpha_i|\tau|))*\frac{2}{\pi}\frac{\sin\omega_b\tau\cos\omega_c\tau}{\tau} \end{aligned} \tag{228}$$

where

$$\frac{2}{\pi}\frac{\sin\omega_b\tau\cos\omega_c\tau}{\tau} \supset B(\omega\,;\,\omega_c, \omega_b)$$

Substituting eq. (228) in eq. (65) the function $f(\tau\,;\,\omega, h)$ is obtained as follows :

$$\begin{aligned} f(\tau\,;\,\omega, h) &= -2a(\tau)\Big[a(\tau)*\Big\{\exp(-h|\omega|\tau)\cos\omega\tau \\ &\cdot\Big(\{\sum_{i,\kappa}(RRS_\psi{}'(\mu^\kappa\alpha_i)\exp(-I\mu^\kappa\alpha_i|\tau|)\sin(R\mu^\kappa\alpha_i|\tau|) \\ &+ IRS_\psi{}'(\mu^\kappa\alpha_i)\exp(-I\mu^\kappa\alpha_i|\tau|)\cos(R\mu^\kappa\alpha_i|\tau|))\}*\frac{2}{\pi}\frac{\sin\omega_b\tau\cos\omega_c\tau}{\tau}\Big)\Big\}\Big] \end{aligned} \tag{229}$$

in which the first asterisk denotes the convolution integral associated with the finite time domain $[0, \tau]$ and the last one denotes the convolution integral associated with the infinite time domain $(-\infty, \infty)$.

Since the convolution integral associated with the infinite time domain is not suitable for the numerical evaluation it is desirable to replace this convolution by the definite integral defined in the finite domain. Denoting the convolution integrals associated with the infinite time domain which appear in eq. (229) as

$$I_s(\tau\,;\,\mu^\kappa\alpha_i) = \frac{2}{\pi}\frac{\sin\omega_b\tau\cos\omega_c\tau}{\tau}*\exp(-I\mu^\kappa\alpha_i|\tau|)\sin(R\mu^\kappa\alpha_i|\tau|) \tag{230}$$

$$I_c(\tau\,;\,\mu^\kappa\alpha_i) = \frac{2}{\pi}\frac{\sin\omega_b\tau\cos\omega_c\tau}{\tau}*\exp(-I\mu^\kappa\alpha_i|\tau|)\cos(R\mu^\kappa\alpha_i|\tau|) \tag{231}$$

and by making use of the integral formulae,

$$\begin{aligned} \int_{-\infty}^{\infty}\exp(-a|\tau|)\sin b|\tau|\exp(-j\mu\tau)d\tau &= I\int_0^\infty(\exp\{-(a-j(b-\mu))\tau\} \\ &\qquad -\exp\{-(a+j(b+\mu))\tau\})d\tau \\ &= \frac{2b(a^2+b^2-\mu^2)}{\{a^2+(b-\mu)^2\}\{a^2+(b+\mu)^2\}} \end{aligned} \tag{232}$$

$$\begin{aligned} \int_{-\infty}^{\infty}\exp(-a|\tau|)\cos b|\tau|\exp(-j\mu\tau)d\tau &= R\int_0^\infty(\exp\{-(a-j(b-\mu))\tau\} \\ &\qquad +\exp\{-(a+j(b+\mu))\tau\})d\tau \\ &= \frac{2a(a^2+b^2+\mu^2)}{\{a^2+(b-\mu)^2\}\{a^2+(b+\mu)^2\}} \end{aligned} \tag{233}$$

eqs. (230) and (231) can be rewritten by the following definite integrals defined in the finite frequency domain, respectively.

$$I_s(\tau\,;\,\mu^\kappa\alpha_i)=-\frac{2R(\mu^\kappa\alpha_i)}{\pi}\int_{\omega_c-\omega_b}^{\omega_c+\omega_b}\frac{(\mu^2-|\mu^\kappa\alpha_i|^2)\cos\mu\tau d\mu}{\mu^4+2(I^2(\mu^\kappa\alpha_i)-R^2(\mu^\kappa\alpha_i))\mu^2+|\mu^\kappa\alpha_i|^4} \tag{234}$$

$$I_c(\tau\,;\,\mu^\kappa\alpha_i)=\frac{2I(\mu^\kappa\alpha_i)}{\pi}\int_{\omega_c-\omega_b}^{\omega_c+\omega_b}\frac{(\mu^2+|\mu^\kappa\alpha_i|^2)\cos\mu\tau d\mu}{\mu^4+2(I^2(\mu^\kappa\alpha_i)-R^2(\mu^\kappa\alpha_i))\mu^2+|\mu^\kappa\alpha_i|^4} \tag{235}$$

Hence the function $f(\tau\,;\,\omega,h)$ is expressed in terms of the quantities given by eqs. (234) and (235) as follows:

$$\begin{aligned}f(\tau\,;\,\omega,h)=&-2a(\tau)[a(\tau)*[\exp(-h|\omega|\tau)\cos\omega\tau\\&\cdot\{\sum_{i,\kappa}(RRS_\psi'(\mu^\kappa\alpha_i)I_s(\tau\,;\,\mu^\kappa\alpha_i)+IRS_\psi'(\mu^\kappa\alpha_i)I_c(\tau\,;\,\mu^\kappa\alpha_i))\}]]\end{aligned} \tag{236}$$

Case B, $\delta=0$

In this case it is convenient to express $S_\psi'(\omega)$ as the sum of a constant and the frequency function, the order of which is given by a positive integer, namely

$$S_\psi'(\omega)=c^2+c^2S_\psi''(\omega),\qquad \text{i.e.,}\qquad S_\psi''(\omega)=\frac{S_\psi'(\omega)}{c^2}-1 \tag{237}$$

in which $S_\psi''(\omega)$ has the order $0(\omega^{-\delta})$, $\delta\geqq 1$ when $|\omega|$ tends to infinity.

If all the poles of $S_\psi''(\omega)$ are simple and they are again denoted by $\{\mu^\kappa\alpha_i\}$ and $\{\mu^\kappa\alpha_i^*\}$ in the upper and the lower half-plane respectively, the auto-correlation function $R_\psi(\tau)$ is expressed as follows:

$$\begin{aligned}R_\psi(\tau)&=\frac{1}{2\pi}\int_{-\infty}^{\infty}S_\psi(\mu)\exp(j\mu\tau)d\mu=\frac{1}{2\pi}\int_{-\infty}^{\infty}B(\mu;\,\omega_c,\,\omega_b)(c^2+c^2S_\psi''(\omega))\exp(j\mu\tau)d\mu\\&=c^2[\delta(\tau)-\sum_{i,\kappa}(RRS_\psi''(\mu^\kappa\alpha_i)\exp(-I\mu^\kappa\alpha_i|\tau|)\sin(R\mu^\kappa\alpha_i|\tau|)\\&\quad+IRS_\psi''(\mu^\kappa\alpha_i)\exp(-I\mu^\kappa\alpha_i|\tau|)\cos(R\mu^\kappa\alpha_i|\tau|))]*\frac{2}{\pi}\frac{\sin\omega_b\tau\cos\omega_c\tau}{\tau}\end{aligned} \tag{238}$$

Substituting eq. (238) in eq. (65) and by making use of the notations given by eqs. (234) and (235), the function $f(\tau\,;\,\omega,h)$ is expressed in the following form:

$$\begin{aligned}f(\tau\,;\,\omega,h)=&\,c^2a^2(\tau)-2c^2a(\tau)[a(\tau)*[\exp(-h|\omega|\tau)\cos\omega\tau\\&\cdot\{\sum_{i,\kappa}(RRS_\psi''(\mu^\kappa\alpha_i)I_s(\tau\,;\,\mu^\kappa\alpha_i)+IRS_\psi''(\mu^\kappa\alpha_i)I_c(\tau\,;\,\mu^\kappa\alpha_i))\}]]\end{aligned} \tag{239}$$

In particular, for the case where both ω_b and ω_c tend to infinity, the band-limiting operator defined by eq. (224) reduces to

$$B(\omega\,;\,\omega_c,\,\omega_b)=1 \tag{240}$$

hence, eq. (225) becomes

$$S_\psi(\omega)=S_\psi'(\omega) \tag{241}$$

In this special case the quantities defined by eqs. (230) and (231) are expressed as

$$I_s(\tau\,;\,\mu^\kappa\alpha_i)=\exp(-I\mu^\kappa\alpha_i|\tau|)\sin(R\mu^\kappa\alpha_i|\tau|) \tag{242}$$

$$I_c(\tau\,;\,\mu^\kappa\alpha_i)=\exp(-I\mu^\kappa\alpha_i|\tau|)\cos(R\mu^\kappa\alpha_i|\tau|) \tag{243}$$

by making use of the following formula:

$$\lim_{\omega_b=\omega_c\to\infty} \frac{2}{\pi} \frac{\sin \omega_b\tau \cos \omega_c\tau}{\tau} = \delta(\tau) \tag{244}$$

Hence the function $f(\tau ; \omega, h)$ which is given by eq. (236) or eq. (239) corresponding to case A or case B is expressed as follows:

Case A, $\delta \geqq 1$

$$\begin{aligned} f(\tau ; \omega, h) = & -2a(\tau)[a(\tau)*[\exp(-h|\omega|\tau)\cos \omega\tau \\ & \cdot \{\sum_{i,\kappa}(\boldsymbol{R}RS_\phi(\mu^\kappa\alpha_i)\exp(-\boldsymbol{I}\mu^\kappa\alpha_i\tau)\sin(\boldsymbol{R}\mu^\kappa\alpha_i\tau) \\ & +\boldsymbol{I}RS_\phi(\mu^\kappa\alpha_i)\exp(-\boldsymbol{I}\mu^\kappa\alpha_i\tau)\cos(\boldsymbol{R}\mu^\kappa\alpha_i\tau))\}]] \end{aligned} \tag{245}$$

Case B, $\delta=0$

$$\begin{aligned} f(\tau ; \omega, h) = & c^2a^2(\tau)-2c^2a(\tau)[a(\tau)*[\exp(-h|\omega|\tau)\cos \omega\tau \\ & \cdot \{\sum_{i,\kappa}(\boldsymbol{R}RS_\phi''(\mu^\kappa\alpha_i)\exp(-\boldsymbol{I}\mu^\kappa\alpha_i\tau)\sin(\boldsymbol{R}\mu^\kappa\alpha_i\tau) \\ & +\boldsymbol{I}RS_\phi''(\mu^\kappa\alpha_i)\exp(-\boldsymbol{I}\mu^\kappa\alpha_i\tau)\cos(\boldsymbol{R}\mu^\kappa\alpha_i\tau))\}]] \end{aligned} \tag{246}$$

In the most special case where $B(\omega ; \omega_c, \omega_b)=1$ and $S_\phi'(\omega)=c^2$, $S_\phi'(\omega)$ defined by eq. (237) becomes identically zero and the following simple expression for $f(\tau ; \omega, h)$ is obtained directly from eq. (246):

$$f(\tau ; \omega, h)=c^2a^2(\tau) \tag{247}$$

The analytical expression of the energy spectral density $S_{E\xi}(\tau ; \omega, h, \tau_d)$ of the modified quasi-stationary random process in the time domain $[0, \tau_d]$ is obtained by substituting the function $f(\tau ; \omega, h)$ given by one of the eqs. (236), (239), (245), (246) and (247) in eq. (63). In general, to find the maximum value of the energy spectral density $S_{E\xi}(\tau_{max} ; \omega, h, \tau_d)$, $0<\tau_{max}\leqq\tau_d$, it may be necessary to carry out the numerical evaluation of eq. (63).

If an approximate value of the maximum τ_{max} is found in the time domain $(0, \tau_d]$, the following successive approximation procedure may be adopted in obtaining the more accurate value of the maximum of the energy spectral density. As in the preceding subsection, if τ_{max} coincides with τ_d the power spectral density given by eq. (64) together with one of eqs. (236), (239), (245), (246) and (247) should be non-negative at τ_d. On the other hand, if the maximum τ_{max} exists in the time domain $(0, \tau_d)$ it should be a zero of the power spectral density. Then from eq. (64) the following equation is obtained:

$$2h|\omega|\int_0^{\tau_{max}}\exp\{-2h|\omega|(\tau_{max}-\kappa)\}f(\kappa ; \omega, h)d\kappa=f(\tau_{max} ; \omega, h) \tag{248}$$

By making use of eq. (63), the above equation is rewritten as follows:

$$2h|\omega|S_{E\xi}(\tau_{max} ; \omega, h, \tau_d)=f(\tau_{max} ; \omega, h) \tag{249}$$

Hence the following successive approximation procedure to determine the maximum τ_{max} of the energy spectral density is obtained:

$$\tau^{(n+1)}=\tau^{(n)}+\Delta\tau^{(n)}, \qquad n=1, 2, \cdots\cdots \tag{250}$$

$$\Delta\tau^{(n)}=-\frac{S_{H\xi}(\tau^{(n)} ; \omega, h, \tau_d)}{S_{H\xi\tau}{}^{(1)}(\tau^{(n)} ; \omega, h, \tau_d)}=-\frac{S_{H\xi}(\tau^{(n)} ; \omega, h, \tau_d)}{f_\tau{}^{(1)}(\tau^{(n)} ; \omega, h)-2h|\omega|S_{H\xi}(\tau^{(n)} ; \omega, h, \tau_d)}$$

$$=\left[2h|\omega|-\left(\frac{f_\tau^{(1)}(\tau^{(n)};\omega,h)}{f(\tau^{(n)};\omega,h)-2h|\omega|S_{E\xi}(\tau^{(n)};\omega,h,\tau_d)}\right)\right]^{-1}$$

$$=\left[2h|\omega|-\left(\frac{f_\tau^{(1)}(\tau^{(n)};\omega,h)}{f(\tau^{(n)};\omega,h)-2h|\omega|\exp(-2h|\omega|\tau)*f(\tau;\omega,h)|_{\tau=\tau^{(n)}}}\right)\right]^{-1} \tag{251}$$

where

$$f_\tau^{(1)}(\tau;\omega,h)=\left(a(\tau)\frac{f(\tau;\omega,h)}{a(\tau)}\right)_\tau^{(1)}=a_\tau^{(1)}(\tau)\left(\frac{f(\tau;\omega,h)}{a(\tau)}\right)+a(\tau)\left(\frac{f(\tau;\omega,h)}{a(\tau)}\right)_\tau^{(1)} \tag{252}$$

By making use of eq. (65), it is shown that the above equation may be written in terms of the following expressions:

$$\left(\frac{f(\tau;\omega,h)}{a(\tau)}\right)=2a(\tau)*\exp(-h|\omega|\tau)\cos\omega\tau R_\psi(\tau) \tag{253}$$

$$\left(\frac{f(\tau;\omega,h)}{a(\tau)}\right)_\tau^{(1)}=2[a(0)\exp(-h|\omega|\tau)\cos\omega\tau R_\psi(\tau)$$
$$+a_\tau^{(1)}(\tau)*\exp(-h|\omega|\tau)\cos\omega\tau R_\psi(\tau)] \tag{254}$$

By making use of the analytical expression of $R_\psi(\tau)$ given by eq. (236) or eq. (239) as well as eqs. (234) and (235), the above equations may reduce to forms suitable for numerical evaluations.

Particulary for the case of a time-invariant envelope $a(\tau)=1$, the following formulae are obtained;

$$f(\tau;\omega,h)=2\int_0^\tau \exp(-h|\omega|\kappa)\cos\omega\kappa R_\psi(\kappa)d\kappa \tag{255}$$

$$f_\tau^{(1)}(\tau;\omega,h)=2\exp(-h|\omega|\tau)\cos\omega\tau R_\psi(\tau) \tag{256}$$

$$f(\tau;\omega,h)-2h|\omega|\exp(-2h|\omega|\tau)*f(\tau;\omega,h)$$
$$=2\int_0^\tau \exp(-h|\omega|\kappa)\cos\omega\kappa R_\psi(\kappa)d\kappa$$
$$-2h|\omega|\int_0^\tau d\nu_2\exp(-2h|\omega|\nu_2)\int_{-\nu_2}^{\tau-\nu_2}d\nu_1\exp(-h|\omega|\nu_1)\cos\omega\nu_1 R_\psi(\nu_1) \tag{257}$$

then, by substituting eqs. (255)$\sim$(257) in eq. (251), the resultant equation reduces to eq. (222).

As another special case, if $S_\psi(\omega)=c^2$, eq. (248) is written as follows by making use of eq. (247):

$$a^2({}_0\tau_{max})=2h|\omega|\int_0^{{}_0\tau_{max}}\exp\{-2h|\omega|({}_0\tau_{max}-\kappa)\}a^2(\kappa)d\kappa$$
$$=2h|\omega|\int_0^{{}_0\tau_{max}}\exp(-2h|\omega|\kappa)a^2({}_0\tau_{max}-\kappa)d\kappa \tag{258}$$

where subscript 0 means that the quantity with it concerns to the case of the white spectrum $S_\psi(\omega)=c^2$.

By considering $R_\psi(\tau)=c^2\delta(\tau)$ eqs. (253), (254) and (256) become respectively

$$\left(\frac{f(\tau;\omega,h)}{a(\tau)}\right)=c^2a(\tau) \tag{259}$$

$$\left(\frac{f(\tau;\omega,h)}{a(\tau)}\right)_\tau^{(1)}=c^2a_\tau^{(1)}(\tau) \tag{260}$$

$$f_\tau^{(1)}(\tau\,;\,\omega,h)=2c^2a(\tau)a_\tau^{(1)}(\tau) \tag{261}$$

Hence eq. (251) is expressed as follows:

$$\varDelta_0\tau^{(n)}=\left[2h|\omega|-\left(\frac{2a({}_0\tau^{(n)})a_\tau^{(1)}({}_0\tau^{(n)})}{a^2({}_0\tau^{(n)})-2h|\omega|\exp(-2h|\omega|\tau)*a^2(\tau)|_{\tau={}_0\tau^{(n)}}}\right)\right]^{-1} \tag{262}$$

From the above equation, it is found that if the time at which the continuous envelope $a(\tau)$ takes the maximum value is chosen as the first approximation, the first increment of ${}_0\tau_{max}$ is a positive number which is inversely proportional to $h|\omega|$. Hence if the damping parameter h is not zero and the power spectral density $S_\phi(\omega)$ is sufficiently flat and if the envelope $a(\tau)$ is a slowly varying continuous function the second approximation of ${}_0\tau_{max}$ for the white spectrum, which is obtained by starting from the maximum of $a(\tau)$, may be used as the first approximation $\tau^{(1)}$ in the successive approximation procedure for the general case given by eqs. (250) and (251). However, if the damping parameter h is zero eq. (262) reduces to

$$\varDelta_0\tau^{(n)}=-\frac{a({}_0\tau^{(n)})}{2a_\tau^{(1)}({}_0\tau^{(n)})},\qquad h=0 \tag{263}$$

Since it is clearly a contradiction that a set of positive values of $a(\tau)$ and $a_\tau^{(1)}(\tau)$ always gives the negative increment of ${}_0\tau_{max}$, the above equation means that the maximum ${}_0\tau_{max}$ is equal to the end point of random excitations, τ_d for an arbitrary envelope $a(\tau)$ as far as the white spectrum and zero damping parameter are concerned.

In general, it is noted that the convergent value of the above-mentioned successive approximation procedure does not always give the maximum τ_{max} of the energy spectral density in the time interval $(0,\tau_d)$. The stationary condition expressed by eq. (249) is only one of the necessary conditions requisite for τ_{max}. From the exact point of view, in order to assure that the convergent value of the successive approximation procedure starting from an appropriate first approximation gives the true maximum τ_{max}, it is necessary to show not only that the convex condition $S_{H\xi\tau}^{(1)}(\tau_{max}\,;\,\omega,h,\tau_d)<0$ is valid but also that $S_{E\xi}(\tau_{max}\,;\,\omega,h,\tau_d)$ definitely gives the maximum value in the time domain $(0,\tau_d)$ because several maxima may exist in this time domain.

From the above-mentioned aspects, in determining the maximum value of the energy spectral density of the modified quasi-stationary random process, the numerical evaluation of the energy spectral density in the time domain $[0,\tau_d]$ should be preceded and the successive approximation procedure must be considered as the auxiliary means to improve the accuracy of the approximate value of the maximum τ_{max}.

For a simple envelope $a(\tau)$ and power spectral density $S_\phi(\omega)$, the explicit analytical expressions without the integral operators of the energy and power spectral densities of the modified quasi-stationary random process may be obtained. As such an example, the following special case is considered:

$$a(\tau)=\sqrt{\tau}\exp\left(-\frac{\alpha}{2}\tau\right),\qquad S_\phi(\omega)=c^2 \tag{264}$$

By using the first equation of (264), eqs. (247) and (261) are written respectively as follows:

$$f(\tau\,;\,\omega,h)=c^2a^2(\tau)=c^2\tau\exp(-\alpha\tau) \tag{265}$$

$$f_\tau^{(1)}(\tau\,;\,\omega,h)=2c^2a(\tau)a_\tau^{(1)}(\tau)=c^2(1-\alpha\tau)\exp(-\alpha\tau) \tag{266}$$

Substituting eq. (265) in eqs. (63) and (64) the following explicit analytical expressions of the energy and power spectral densities are obtained:

$$\begin{aligned}S_{E\xi}(\tau\,;\,\omega,h,\tau_d)&=c^2\exp(-2h|\omega|\tau)*a^2(\tau)\\&=\frac{c^2}{(\alpha-2h|\omega|)^2}[\exp(-2h|\omega|\tau)-\exp(-\alpha\tau)-(\alpha-2h|\omega|)\tau\exp(-\alpha\tau)],\\&\qquad 0\leqq\tau\leqq\tau_d\end{aligned} \tag{267}$$

$$\begin{aligned}S_{H\xi}(\tau\,;\,\omega,h,\tau_d)&=c^2(\delta(\tau)-2h|\omega|\exp(-2h|\omega|\tau))*a^2(\tau)\\&=\frac{c^2}{(\alpha-2h|\omega|)^2}[\alpha(\alpha-2h|\omega|)\tau\exp(-\alpha\tau)-2h|\omega|(\exp(-2h|\omega|\tau)-\exp(-\alpha\tau))],\\&\qquad 0\leqq\tau<\tau_d\end{aligned} \tag{268}$$

In particular in the case where $h=0$, the power spectral density given by eq. (268) is always positive. Hence the energy spectral density given by eq. (267) is a monotonously increasing function of time in the domain $[0,\tau_d]$, and the maximum ${}_0\tau_{max}$ agrees with τ_d.

As regards the successive approximation procedure of the maximum ${}_0\tau_{max}$, eq. (258) reduces to

$${}_0\tau_{max}\exp(-\alpha_0\tau_{max})=\frac{2h|\omega|}{\alpha(\alpha-2h|\omega|)}(\exp(-2h|\omega|_0\tau_{max})-\exp(-\alpha_0\tau_{max})) \tag{269}$$

and eqs. (250) and (262) are written in the following forms respectively:

$${}_0\tau^{(n+1)}={}_0\tau^{(n)}+\Delta_0\tau^{(n)},\qquad n=1,2,\cdots\cdots \tag{270}$$

$$\Delta_0\tau^{(n)}=\left[2h|\omega|-\left(\frac{(\alpha-2h|\omega|)^2(1-\alpha_0\tau^{(n)})\exp(-\alpha_0\tau^{(n)})}{\alpha(\alpha-2h|\omega|)_0\tau^{(n)}\exp(-\alpha_0\tau^{(n)})-2h|\omega|(\exp(-2h|\omega|_0\tau^{(n)})-\exp(-\alpha_0\tau^{(n)}))}\right)\right]^{-1} \tag{271}$$

Starting from the first approximation which is appropriately chosen as

$$\frac{1}{\alpha}<{}_0\tau^{(1)}\leqq\tau_d \tag{272}$$

the maximum τ_{max} which gives the maximum value of the energy spectral density may be determined as the limiting value of the iterative procedure given by eqs. (270) and (271) because in this case both $S_{E\xi}(\tau\,;\,\omega,h,\tau_d)$ and $S_{H\xi}(\tau\,;\,\omega,h,\tau_d)$ are simple smoothed functions in the time domain $[0,\tau_d)$ as shown in eqs. (267) and (268).

6. Probable expressions of the mean value and the upper and lower limits of response spectra of the quasi-stationary random excitations

In section 4, the formal expressions of the mean value and the upper and lower limits of response spectra of the quasi-stationary random excitations are determined as eq. (154) and eqs. (160) and (161) respectively, in which the quantities λ, $\bar{\lambda}_{ar}$, $V\lambda$, $\bar{\mu}$ and $\underline{\mu}$ remain as unknown functions of ω, h and τ_d as sociated with the original quasi-stationary random excitations $f(\tau)$. In this section the explicit functional forms of these quantities are considered referring to the results of response analyses of a linear system subjected to quasi-stationary random excitations.

If the duration time τ_d of the quasi-stationary random excitations having a slowly varying envelope is sufficiently large, more exactly, if the maximum τ_{max} is sufficiently large compared with the natural period of the linear system $t=\frac{2\pi}{\omega}$ and if the power spectral density $S_\phi(\omega)$ is sufficiently flat over the frequency range considered, the quantities λ and $\bar{\lambda}_{av}$ may be approximately equal and they seem to be weak functions of ω and τ_d. In such a case it is suggested that instead of the functions λ and $\bar{\lambda}_{av}$ the mean functions averaged in the $\omega-\tau_d$ domain considered may be available in eqs. (154), (160) and (161), namely

$$ {}_f\lambda(h) = < {}_f\lambda(\omega, h, \tau_d) >_{\omega,\tau_d} \doteqdot < {}_f\bar{\lambda}_{av}(\omega, h, \tau_d) >_{\omega,\tau_d} = E < {}_f\bar{\lambda}_{T}(\omega, h, \tau_d) >_{\omega,\tau_d} \tag{273} $$

in which the symbol $<A>_a$ denotes the mean value of A with respect to a and subscript f means that the quantity with it depends on the original quasi-stationary random excitations $f(\tau)$.

Since the function ${}_f\lambda(h)$ defined by eq. (273) seems to have the properties,

$$ {}_f\lambda(h) \geqq 1, \qquad {}_f\lambda_h^{(1)}(h) > 0, \qquad {}_f\lambda_h^{(2)}(h) < 0, \qquad h > 0 \tag{274} $$

and it also seems to converge sharply to a constant value in the vicinity of $h=0$, the following probable expression of ${}_f\lambda(h)$ may be obtained;

$$ {}_f\lambda(h) = {}_f\lambda(\infty) - \{ {}_f\lambda(\infty) - {}_f\lambda(0) \} \exp\{ - {}_f\alpha(h) h \} \tag{275} $$

in which the function ${}_f\alpha(h)$ indicates the degree of the convergence of ${}_f\lambda(h)$ and may have the following properties:

$$ {}_f\alpha(h) \gg 1, \qquad {}_f\alpha_h^{(1)}(h) < 0, \qquad {}_f\alpha_h^{(2)}(h) > 0 \tag{276} $$

In addition to the above properties ${}_f\alpha(h)$ seems to converge to a constant value as h increases from zero. Hence a probable expression of ${}_f\alpha(h)$ may be given by

$$ {}_f\alpha(h) = {}_f\alpha(\infty) + \{ {}_f\alpha(0) - {}_f\alpha(\infty) \} \exp(-\beta h), \qquad \beta > 0 \tag{277} $$

in which β is a sufficiently large positive number.

From eqs. (275) and (277) the following relation between ${}_f\lambda(h)$ and ${}_f\alpha(h)$ is obtained:

$$ {}_f\alpha(h) = -\frac{1}{h} \log \frac{ {}_f\lambda(\infty) - {}_f\lambda(h) }{ {}_f\lambda(\infty) - {}_f\lambda(0) } \tag{278} $$

To determine the various constants contained in eqs. (275) and (277) the response analyses of the linear system having the impulsive response $g(\tau)=\exp(-h|\omega|\tau)\sin\omega\tau$ should be made for the pertinent quasi-stationary random excitations and the appropriate sets of parameters (ω, h, τ_d), and the equivalent coefficient λ in eq. (154) should be estimated numerically, based upon the results of the response analyses.

If a large number of random responses are obtainable for each set of parameters (ω, h, τ_d), te equivalent coefficient ${}_f\lambda(\omega, h, \tau_d)$ may be evaluated from the following equation:

$$ {}_f\lambda(\omega, h, \tau_d) = \left(\frac{ES_V(\omega, h, \tau_d)}{\sqrt{S_{E\xi}(\tau_{max};\, \omega, h, \tau_d)}} - \frac{\sqrt{\pi}}{2} \right) \Big/ \sqrt{1 - \frac{\pi}{4}} \tag{279} $$

On the other hand, if the size of the ensemble of random responses is small for each set of parameters (ω, h, τ_d) the equivalent coefficient ${}_f\lambda(\omega, h, \tau_d)$ may be estimated by the following procedure: First, for each pseudo-stationary random time-function $|A_s|$ defined by eq. (144), the maximum normalized random variable

$$ {}_s\bar{\lambda}_T(\omega, h, \tau_d) = \frac{\sup\limits_T(|\dot{A}|_s - E_T|A_s|)}{\sqrt{V_T|A_s|}} \tag{280} $$

and the following two quantities are calculated;

$$ {}_sd_{1T}(\omega, h, \tau_d) = \frac{E_T|A_s|}{\sqrt{E_T|A_s|^2}}, \qquad {}_s\sigma_T(\omega, h, \tau_d) = \frac{\sqrt{V_T|A_s|}}{E_T|A_s|} \tag{281} $$

and, by making use of these quantities the equivalent maximum random variable ${}_f\bar{\lambda}_T(\omega, h, \tau_d)$ associated with the Rayleigh distribution is determined as follows:

$$ {}_f\bar{\lambda}_T(\omega, h, \tau_d) = \left(\frac{{}_s\bar{\lambda}_T(\omega, h, \tau_d)\,{}_s\sigma_T(\omega, h, \tau_d)}{\sigma(0)} \right) \left(\frac{{}_sd_{1T}(\omega, h, \tau_d)}{d_1(0)} \right) + \frac{\left(\frac{{}_sd_{1T}(\omega, h, \tau_d)}{d_1(0)} \right) - 1}{\sigma(0)} \tag{282} $$

where

$$ \sigma(0) = \sqrt{\frac{4-\pi}{\pi}}, \qquad d_1(0) = \frac{\sqrt{\pi}}{2} \tag{283} $$

The correction of the maximum normalized random variable by eq. (282) is due to the deviation of the amplitude probability distribution of $|A_s|$ from the Rayleigh distribution. This deviation may be measured by the parameter ξ defined by eq. (71). The values of $d_1(\xi)$, $d_2(\xi)$ and $\sigma(\xi)$ given by eqs. (88) and (89) for $\xi=0$ which corresponds to the Rayleigh distribution and $\xi=\pm1$ are calculated as follows:

$$ d_1(0) = \frac{\sqrt{\pi}}{2} = 0.886, \qquad d_1(\pm1) = \sqrt{\frac{2\pi}{4+\pi}} = 0.938 $$

$$ d_2(0) = \frac{\sqrt{4-\pi}}{2} = 0.464, \qquad d_2(\pm1) = \sqrt{\frac{4-\pi}{4+\pi}} = 0.352 $$

$$ \sigma(0) = \sqrt{\frac{4-\pi}{\pi}} = 0.524, \qquad \sigma(\pm1) = \sqrt{\frac{4-\pi}{2\pi}} = 0.370 $$

with the aid of these values it is shown that $d_1(\xi)$ is a weak function of ξ compared with $\sigma(\xi)$. Hence, eq. (282) may be expressed approximately by

$$ {}_f\bar{\lambda}_T(\omega, h, \tau_d) = \frac{{}_s\bar{\lambda}_T(\omega, h, \tau_d)\,{}_s\sigma_T(\omega, h, \tau_d)}{\sigma(0)} \tag{284} $$

In the case where the duration time τ_d is fixed and the response analyses are made for various frequency parameters and a series of the discrete values of h_i, $i=1, 2, 3, \cdots\cdots$, the equivalent function ${}_f\lambda(h_i)$ may be determined by substituting eq. (282) or eq. (284) in the last equation of (273) and by averaging with respect to the frequency parameter.

Next, the constants ${}_f\lambda(0)$, ${}_f\lambda(\infty)$, ${}_f\alpha(0)$, ${}_f\alpha(\infty)$ and β which are contained in eqs. (275) and (277) may be determined by the following iterative procedure. Supposing the appropriate values of ${}_f\lambda(0)$ and ${}_f\lambda(\infty)$ from the series ${}_f\lambda(h_i)$ and by substituting these values in eq. (278) the series of the values ${}_f\alpha(h_i)$, $i=1, 2, 3, \cdots\cdots$, are obtained. Then, selecting three arbitrary points $i=l, m, n$ the exponent β_{lmn} in eq. (277) is calculated from the following equation:

$$\frac{\alpha(h_l)-\alpha(h_n)}{\alpha(h_l)-\alpha(h_m)}=\frac{1-\exp\{-\beta(h_n-h_l)\}}{1-\exp\{-\beta(h_m-h_l)\}} \tag{285}$$

And, for several sets of two points $i=m, n$, ${}_f\alpha(0)_{mn}$ and ${}_f\alpha(\infty)_{mn}$ are determined from the following simultaneous equation:

$$\{1-\exp(-\beta h_i)\}{}_f\alpha(\infty)+\exp(-\beta h_i){}_f\alpha(0)=\alpha(h_i), \qquad i=m, n \tag{286}$$

Thus the constants β, ${}_f\alpha(0)$ and ${}_f\alpha(\infty)$ are determined through an averaging operation as follows:

$$\beta=\langle\beta_{lmn}\rangle_{lmn}, \qquad {}_f\alpha(0)=\langle{}_f\alpha(0)_{mn}\rangle_{mn}, \qquad {}_f\alpha(\infty)=\langle{}_f\alpha(\infty)_{mn}\rangle_{mn} \tag{287}$$

in which subscript *lmn* or *mn* shows the quantity, with it being dependent on the set of points selected. By substituting the quantities given by eq. (287) in eq. (277) the new series of values ${}_f\alpha(h_i)$, $i=1, 2, 3, \cdots\cdots$, are obtained. By making use of these values and the previously determined values of ${}_f\lambda(h_i)$ for several sets of two points $i=m, n$, a set of ${}_f\lambda(0)_{mn}$ and ${}_f\lambda(\infty)_{mn}$ is obtained from the following simultaneous equations:

$$[1-\exp\{-{}_f\alpha(h_i)h_i\}]{}_f\lambda(\infty)+\exp\{-{}_f\alpha(h_i)h_i\}{}_f\lambda(0)={}_f\lambda(h_i), \qquad i=m, n \tag{288}$$

Then the constants ${}_f\lambda(0)$ and ${}_f\lambda(\infty)$ are determined by averaging ${}_f\lambda(0)_{mn}$ and ${}_f\lambda(\infty)_{mn}$ respectively as follows:

$${}_f\lambda(0)=\langle{}_f\lambda(0)_{mn}\rangle_{mn}, \qquad {}_f\lambda(\infty)=\langle{}_f\lambda(\infty)_{mn}\rangle_{mn} \tag{289}$$

Otherwise, they may be determined by the following least square method:

$$\varepsilon^2({}_f\lambda(0), {}_f\lambda(\infty))=\sum_i[{}_f\lambda(\infty)-\{{}_f\lambda(\infty)-{}_f\lambda(0)\}\exp\{-{}_f\alpha(h_i)h_i\}-{}_f\lambda(h_i)]^2 \tag{290}$$

$$\frac{\partial}{\partial_f\lambda(0)}\varepsilon^2({}_f\lambda(0), {}_f\lambda(\infty))=0, \qquad \frac{\partial}{\partial_f\lambda(\infty)}\varepsilon^2({}_f\lambda(0), {}_f\lambda(\infty))=0$$

If the values thus obtained of ${}_f\lambda(0)$ and ${}_f\lambda(\infty)$ are different from the previously assumed values beyond the prescribed allowable error the above-mentioned procedure must be repeated until the error is reduced within the allowable value.

In the following, to examine the above-mentioned procedure for determing ${}_f\lambda(h)$ a numerical example obtained by the simulation method is shown. The quasi-stationary random excitations considered are characterized by the time-invariant envelope $a(\tau)=1$, the duration time $\tau_d=30$ sec and the following power spectral density:

$$S_\psi(\omega)=B(\omega\,;\,\omega_c, \omega_b)S_\psi'(\omega) \tag{291}$$

$$\frac{S_{\psi}'(\omega)}{\omega^2}=\frac{\left\{1+\left(\frac{\omega}{\omega_1}\right)^4\right\}\left\{1+\left(\frac{\omega}{\omega_2}\right)^2\right\}\left\{1+\left(\frac{\omega}{\omega_3}\right)^2\right\}\left\{1+\left(\frac{\omega}{\omega_4}\right)^8\right\}}{\left\{1+\left(\frac{\omega}{\omega_1}\right)^2\right\}\left\{1+\left(\frac{\omega}{\omega_2}\right)^4\right\}\left\{1+\left(\frac{\omega}{\omega_3}\right)^8\right\}\left\{1+\left(\frac{\omega}{\omega_4}\right)^2\right\}} \tag{292}$$

in which

$$\begin{aligned}&\omega_c=18.2\pi, \quad \omega_b=17.3\pi, \quad (\text{rad/sec})\\ &\omega_1=2\pi, \quad \omega_2=3.6\pi, \quad \omega_3=10.4\pi, \quad \omega_4=22.0\pi, \quad (\text{rad/sec})\end{aligned} \tag{293}$$

From each sample random response of the single-degree-of-freedom systems having the natural frequency $\omega=10\pi$ rad/sec and various values of damping parameter h, the amplitude probability distribution of the pseudo-stationary process $|A_s|$ is evaluated by using the values of the peak amplitude at 200 points contained in the time domain $T=\left[\frac{\tau_d}{3}, \tau_d\right]=[10 \text{ sec}, 30 \text{ sec}]$. By using eqs. (278), (280), (281) and (284) the following results are obtained:

h	0	0.005	0.01	0.02	0.05
${}_s\sigma_T$	0.289	0.469	0.473	0.497	0.521
${}_s\lambda_T$	1.76	1.73	2.03	2.33	2.53
${}_f\lambda_T$	0.971	1.55	1.84	2.21	2.51
${}_f\bar{\alpha}$	—	65.9	54.4	46.6	27.9

The values in the last row of the above table are evaluated from eq. (278) by supposing ${}_f\lambda(0)=0.971$ and ${}_f\lambda(\infty)=3.000$. From the values of ${}_s\sigma_T$ it is found that the amplitude probability distribution of the pseudo-stationary random process $|A_s|$ converges rapidly to the Rayleigh distribution as the damping parameter h increases from zero. However, in the case of $h=0$ the original random response $|J_\xi|$ or $|A_\xi|$ seems to belong to a strongly divergent process. As shown in the table the value of ${}_s\sigma_T$ is considerably smaller than $\sigma(0)=\sqrt{(4-\pi)/\pi}=0.524$ and it is indicated from eq. (89) that the corresponding value of ξ is larger than unity. Therefore it is suggested that the correction given by eq. (282) or eq. (284) is necessary for the case of a sufficiently small value of the damping parameter. The corrected values ${}_f\bar{\lambda}_T$ are shown in the third row of the table.

Even though the values shown in the table are based on the data which are obtained from a sample random time-function prescribed by $\omega=10\pi$ rad/sec and $\tau_d=30$ sec for each damping parameter, they show a sufficiently smoothed tendency for the equivalent function ${}_f\lambda(h)$ which gives the mean value of response spectra of the quasi-stationary random excitations. As a rule, to definitely determine the function ${}_f\lambda(h)$ expressed by eq. (275), the averaged value ${}_f\lambda(\omega, h, \tau_d)$ must be evaluated from an ensemble of the random variables ${}_f\bar{\lambda}_T(\omega, h, \tau_d)$ for each set of parameters (ω, h, τ_d) and again the averaging operation with respect to ω and τ_d is required as shown in eq. (273). In this section, however, for the purpose of obtaining the probable expressions of the mean value and the upper and lower limits of response spectra of the quasi-stationary random process, the applicability of eqs. (275) and (277) to such expressions is examined by determining the unknown constants contained in eqs. (275) and (277) by using the following rough, short-cut procedure:

By using the values of ${}_f\bar{\alpha}$ given in the last row of the table the function ${}_f\alpha(h)$ is determined in the following form according to eqs. (285) and (286).

$$ {}_f\alpha(h)=16.5+58.0\exp(-32.0h) \tag{294}$$

And by making use of the above equation and the values of ${}_f\bar{\lambda}_T$ for $h=0.005$ and $h=0.05$, which are given in the third row of the table, the values of $\lambda_f(0)$ and ${}_f\lambda(\infty)$ are calculated from eq. (288) as 0.964 and 3.01, respectively. Then, by substituting these values in eq. (275) the following expression of ${}_f\lambda(h)$ is obtained:

$$ {}_f\lambda(h)=3.01-2.05\exp\{-{}_f\alpha(h)h\} \tag{295}$$

On the other hand, defining the function ${}_fq(h)$ by

$$\begin{aligned} {}_fq(h)&=\left\langle\frac{ES_V(\omega,h,\tau_d)}{ES_V(\omega,0,\tau_d)}\sqrt{\frac{S_{E\xi}(\tau_{max};\omega,0,\tau_d)}{S_{E\xi}(\tau_{max};\omega,h,\tau_d)}}\right\rangle_{\omega,\tau_d} \\ &=\frac{\frac{\sqrt{\pi}}{2}+{}_f\lambda(h)\sqrt{1-\frac{\pi}{4}}}{\frac{\sqrt{\pi}}{2}+{}_f\lambda(0)\sqrt{1-\frac{\pi}{4}}}, \qquad q(0)=1 \end{aligned} \tag{296}$$

and by using eq. (295) in the numerator of the right-hand side the above equation can be written as follows:

$$ {}_fq(h)=1.71-0.71\exp\{-{}_f\alpha(h)h\} \tag{297}$$

Although the function ${}_f\lambda(h)$ given by eq. (295) is determined by means of a simulation method using a special quasi-stationary random excitations and rather rough estimation procedure, the value ${}_fq(\infty)=1.71$ calculated from eq. (297) seems to be satistactorily close to the value $q(\infty)=1.56$ evaluated from the formula which is presented by G. W. Housner and P. C. Jennings based upon the response spectra of past strong earthquakes. Also it is found that the multiplication factor for $h=0$, which is determined as $\sqrt{\frac{\pi}{2}}+{}_f\lambda(0)\sqrt{\frac{(4-\pi)}{4}}=1.33$ by using eqs. (154) and (295), is little larger than the theoretical value 1.174, which is presented by E. Rosenblueth for the case of zero damping and whith noise excitations. Hence, from a practical point of view, it is permissible to suppose $\lambda(0)=1$ and $\lambda(\infty)=3$ which correspond to the multiplication factor for zero damping $=1.35$ and $q(\infty)=1.69$, respectively. As an example, substituting these values in eq. (275) and by considering eq. (294) the following probable expression to estimate the mean values of the response spectra of the earthquake excitations may be obtained:

$$ES_V(\omega,h,\tau_d)=\left(\frac{\sqrt{\pi}}{2}+\lambda(h)\sqrt{1-\frac{\pi}{4}}\right)\sqrt{S_{E\xi}(\tau_{max};\omega,h,\tau_d)} \tag{298}$$

where

$$\lambda(h)=3-2\exp\{-\alpha(h)h\},\qquad \alpha(h)=15+60\exp\{-30h\},\qquad 1\gg h>0 \tag{299}$$

As regards the maximum value of the energy spectral density of the modified quasi-stationary random process contained in eq. (298), eq. (192) may be available in the case where $1\gg h>0$, because the strong earthquake excitations usually have a slowly varying envelope having a sufficiently large duration time and flat spectral characteristics over a wide frequency range[4),11)] and the elastic structural system may have a critical damping ratio which is sufficiently small compared to unity but not zero.

It is noticed that eqs. (298) and (299) are derived by assuming that the function ${}_f\lambda(\omega, h, \tau_d)$ is a weak function of ω and τ_d. This assumption may be valid in the case where the quantity $2\pi/\tau_{max}$ is sufficiently small compared with the lower limit of the frequency band of $S_Y(\omega, h, \tau_d)$ considered and the power spectral density $S_\psi(\omega)$ is sufficiently flat over this frequency band. On the contrary, if $2\pi/\tau_{max}$ is not sufficiently small compared with the frequency parameter considered or the power spectral density $S_\psi(\omega)$ is not flat over the frequency band, the function ${}_f\lambda(\omega, h, \tau_d)$ seems to be a comparatively strong function not only of h but also of ω and τ_d.

For instance in the case where the power spectral density $S_\psi(\omega)$ is sufficiently flat over the frequency band but $\tau_{max}\,\omega$ is not always sufficiently large in this frequency band, the probable expression of the function ${}_f\lambda(\omega, h, \tau_d)$ may be given by the following form:

$$ {}_f\lambda(\omega, h, \tau_d) \equiv {}_f\lambda(\omega, h, \tau_{max}) = {}_f\lambda(\infty) - \{{}_f\lambda(\infty) - {}_f\lambda(0)\}\exp(-{}_f\alpha(\omega, h, \tau_{max})h) \tag{300} $$

$$ {}_f\alpha(\omega, h, \tau_{max}) = (1-\exp(-a\omega\tau_{max}))[{}_f\alpha(\infty) - \{{}_f\alpha(\infty) - {}_f\alpha(0)\}\exp(-\beta h)] \tag{301} $$

in which

$$ \begin{aligned} &{}_f\lambda(\infty) > {}_f\lambda(0) > 0, \qquad {}_f\alpha(0) > {}_f\alpha(\infty) > 0 \\ &a > 0, \qquad \beta > 0 \end{aligned} \tag{302} $$

In eqs. (300) and (301) τ_{max} is a function of ω, h and τ_d which depends on the original quasi-stationary random process $f(\tau)$. Since ${}_f\lambda(\omega, h, \tau_d)$ seems to depend more explicity on τ_{max} than τ_d, the notation ${}_f\lambda(\omega, h, \tau_{max})$ is used instead of ${}_f\lambda(\omega, h, \tau_d)$ in eq. (300).

As shown in eq. (300) the function ${}_f\lambda(\omega, h, \tau_{max})$ is symmetrical with respect to ω and τ_{max}. In particular, for the case of $h=0$ or $h=\infty$, the function ${}_f\lambda(\omega, h, \tau_{max})$ takes a constant value which is given by ${}_f\lambda(0)$ or ${}_f\lambda(\infty)$ in eq. (300). Also, for both cases of $\omega=0$ and $\tau_{max}=0$, the function ${}_f\lambda(\omega, h, \tau_{max})$ gives the constant value ${}_f\lambda(0)$.

From eqs. (300), (301) and (302) the following inequalities which seem to be valid in usual cases are derived:

$$ \begin{aligned} &{}_f\lambda(\omega, h, \tau_{max}) > 0, \\ &{}_f\lambda_\omega{}^{(1)}(\omega, h, \tau_{max}) > 0, \qquad {}_f\lambda_{\tau_{max}}{}^{(1)}(\omega, h, \tau_{max}) > 0 \\ &{}_f\lambda_h{}^{(1)}(\omega, h, \tau_{max}) > 0 \quad \text{if} \quad {}_f\alpha(\omega, h, \tau_{max}) + {}_f\alpha_h{}^{(1)}(\omega, h, \tau_{max})h > 0 \end{aligned} \tag{303} $$

$$ \begin{aligned} &{}_f\alpha(\omega, h, \tau_{max}) > 0, \\ &{}_f\alpha_\omega{}^{(1)}(\omega, h, \tau_{max}) > 0, \qquad {}_f\alpha_{\tau_{max}}{}^{(1)}(\omega, h, \tau_{max}) > 0, \qquad {}_f\alpha_h{}^{(1)}(\omega, h, \tau_{max}) < 0 \\ &{}_f\alpha_\omega{}^{(1)}{}_{\tau_{max}}{}^{(1)}(\omega, h, \tau_{max}) < 0, \quad {}_f\alpha_\omega{}^{(1)}{}_h{}^{(1)}(\omega, h, \tau_{max}) < 0, \quad {}_f\alpha_h{}^{(1)}{}_{\tau_{max}}{}^{(1)}(\omega, h, \tau_{max}) < 0 \\ &{}_f\alpha_\omega{}^{(2)}(\omega, h, \tau_{max}) < 0, \qquad {}_f\alpha_{\tau_{max}}{}^{(2)}(\omega, h, \tau_{max}) < 0, \qquad {}_f\alpha_h{}^{(2)}(\omega, h, \tau_{max}) > 0 \end{aligned} \tag{304} $$

To find the necessary and sufficient condition of the conditional inequality attached to the inequality ${}_f\lambda_h{}^{(1)}(\omega, h, \tau_{max}) > 0$, the following function and its derivative are considered:

$$
\begin{aligned}
{}_f\gamma(\omega, h, \tau_{max}) &= {}_f\alpha(\omega, h, \tau_{max}) + {}_f\alpha_h^{(1)}(\omega, h, \tau_{max})h \\
&= (1-\exp(-a\omega\tau_{max}))[{}_f\alpha(\infty) - \{{}_f\alpha(\infty) - {}_f\alpha(0)\}(1-\beta h)\exp(-\beta h)]
\end{aligned}
\tag{305}
$$

$$
{}_f\gamma_h^{(1)}(\omega, h, \tau_{max}) = (1-\exp(-a\omega\tau_{max}))\{{}_f\alpha(\infty) - {}_f\alpha(0)\}(2-\beta h)\beta\exp(-\beta h) \tag{306}
$$

From eq. (306) the following relations are obtained:

$$
\begin{aligned}
&{}_f\gamma_h^{(1)}(\omega, h, \tau_{max}) \leqq 0 \quad \text{for} \quad h \leqq \frac{2}{\beta} \\
&{}_f\gamma_h^{(1)}(\omega, h, \tau_{max}) > 0 \quad \text{for} \quad h > \frac{2}{\beta}
\end{aligned}
\tag{307}
$$

Since $h = 2/\beta$ is the mimimum of the function ${}_f\gamma(\omega, h, \tau_{max})$ for all ω and τ_{max} the necessary and sufficient condition in order that the function ${}_f\gamma(\omega, h, \tau_{max})$ is non-negative for all ω, h and τ_{max} is given by

$$
{}_f\alpha(\infty) > {}_f\alpha(0)\frac{\exp(-2)}{1+\exp(-2)} \tag{308}
$$

On the other hand, since the mean value of the velocity response spectra $ES_V(\omega, h, \tau_d)$ may be a monotonously decreasing function of the damping parameter h the validity of the following inequality which is obtained by differentaiting eq. (154) with respect to h may be required in usual cases:

$$
\frac{ES_{Vh}^{(1)}(\omega, h, \tau_d)}{ES_V(\omega, h, \tau_d)} = \frac{{}_f\lambda_h^{(1)}(\omega, h, \tau_{max})\sqrt{1-\frac{\pi}{4}}}{\frac{\sqrt{\pi}}{2} + {}_f\lambda(\omega, h, \tau_{max})\sqrt{1-\frac{\pi}{4}}} + \frac{S_{E\xi h}^{(1)}(\tau_{max}\,;\,\omega, h, \tau_d)}{2S_{E\xi}(\tau_{max}\,;\,\omega, h, \tau_d)} < 0 \tag{309}
$$

In particular in the case where $a(\tau) = 1$ and $S_\phi(\omega) = c^2$ the maximum τ_{max} is regarded approximately as τ_d. Then the first and second terms in eq. (309) are expressed respectively as follows by making use of eqs. (182), (183), (300) and (301):

$$
\begin{aligned}
&\frac{{}_f\lambda_h^{(1)}(\omega, h, \tau_{max})\sqrt{1-\frac{\pi}{4}}}{\frac{\sqrt{\pi}}{2} + {}_f\lambda(\omega, h, \tau_{max})\sqrt{1-\frac{\pi}{4}}} \\
&= \Big[\{{}_f\lambda(\infty) - {}_f\lambda(0)\}\{1-\exp(-a\omega\tau_d)\}[{}_f\alpha(\infty) - \{{}_f\alpha(\infty) - {}_f\alpha(0)\}(1-\beta h)\exp(-\beta h)] \\
&\quad \cdot \exp\{-(1-\exp(-a\omega\tau_d))[{}_f\alpha(\infty) - \{{}_f\alpha(\infty) - {}_f\alpha(0)\}\exp(-\beta h)]h\}\sqrt{1-\frac{\pi}{4}}\Big] \\
&\quad \Big/ \Big[\frac{\sqrt{\pi}}{2} + [{}_f\lambda(\infty) - \{{}_f\lambda(\infty) - {}_f\lambda(0)\}\exp\{-(1-\exp(-a\omega\tau_d)) \\
&\quad \cdot [{}_f\alpha(\infty) - \{{}_f\alpha(\infty) - {}_f\alpha(0)\}\exp(-\beta h)]h\}]\sqrt{1-\frac{\pi}{4}}\Big]
\end{aligned}
\tag{310}
$$

and

$$
\begin{aligned}
\frac{S_{E\xi h}^{(1)}(\tau_d\,;\,\omega, h, \tau_d)}{2S_{E\xi}(\tau_d\,;\,\omega, h, \tau_d)} &= \frac{(1+2h|\omega|\tau_d)\exp(-2h|\omega|\tau_d) - 1}{2h(1-\exp(-2h|\omega|\tau_d))} \\
&= \frac{|\omega|\tau_d\exp(-2h|\omega|\tau_d)}{1-\exp(-2h|\omega|\tau_d)} - \frac{1}{2h}, \qquad h > 0
\end{aligned}
\tag{311}
$$

$$
\frac{S_{E\xi h}^{(1)}(\tau_d\,;\,\omega, 0, \tau_d)}{2S_{E\xi}(\tau_d\,;\,\omega, 0, \tau_d)} = -\frac{|\omega|\tau_d}{2}, \qquad h = 0 \tag{312}
$$

Particularly considering the case of $h=0$ eq. (309) is written as follows:

$$\frac{1-\exp(-a\omega\tau_d)}{\omega\tau_d}<\frac{\frac{\sqrt{\pi}}{2}+{}_f\lambda(0)\sqrt{1-\frac{\pi}{4}}}{2\{{}_f\lambda(\infty)-{}_f\lambda(0)\}{}_f\alpha(0)\sqrt{1-\frac{\pi}{4}}} \tag{313}$$

In order for the above inequality to be valid for all $\omega\tau_d$ the constant a should satisfy the following inequality:

$$a<\frac{\frac{\sqrt{\pi}}{2}+{}_f\lambda(0)\sqrt{1-\frac{\pi}{4}}}{2\{{}_f\lambda(\infty)-{}_f\lambda(0)\}{}_f\alpha(0)\sqrt{1-\frac{\pi}{4}}} \tag{314}$$

Since for a comparatively large value of $\omega\tau_{max}$ the factor $(1-\exp(-a\omega\tau_{max}))$ contained in eq. (301) may scarcely affect ${}_f\lambda(\omega, h, \tau_{max})$ given by eq. (300) the values of ${}_f\alpha(0)$, ${}_f\alpha(\infty)$, ${}_f\lambda(0)$ and ${}_f\lambda(\infty)$ which are previously determined for large $\omega\tau_{max}$ may also be applicable to eqs. (300) and (301). Then by substituting ${}_f\alpha(0)=75$, ${}_f\alpha(\infty)=15$, ${}_f\lambda(0)=1$ and ${}_f\lambda(\infty)=3$ in eq. (314) the probable upper limit of the constant a may be determined as 0.0097.

Finally, in connexion with the upper and lower limits of response spectra of the quasi-stationary random excitations which are expressed as in eqs. (160) and (161) respectively, both quantities $\bar{\mu}$ and $\underline{\mu}$ which are defined by eqs. (162) and (163) are generally the functions of ω, h and τ_{max} and they also depend on the statistical properties of the original quasi-stationary random process $f(\tau)$. Usually the ratio of the standard deviation $\sqrt{VS_V}$ to the mean value ES_V, which corresponds to the ratio $\sqrt{V\bar{\lambda}}/(\sqrt{\pi/(4-\pi)}+E\bar{\lambda})$, is an exponentially decreasing function of h and both $\bar{\mu}\sqrt{VS_V}/ES_V$ and $-\underline{\mu}\sqrt{VS_V}/ES_V$ are also the decreasing functions of h though $\bar{\mu}$ and $-\underline{\mu}$ may have the tendency of increasing as h increases.

If the amplitude probability distribution of $\bar{\lambda}$ is assumed to be approximately symmetrical about the mean value $E\bar{\lambda}$, both quantities $\bar{\mu}\sqrt{V\bar{\lambda}}/(\sqrt{\pi/(4-\pi)}+E\bar{\lambda})$ and $-\underline{\mu}\sqrt{V\bar{\lambda}}/(\sqrt{\pi/(4-\pi)}+E\bar{\lambda})$ may be expressed as follows:

$$\begin{aligned}{}_f\delta(\omega, h, \tau_{max})&=\bar{\mu}\sqrt{V\bar{\lambda}}\Big/\Big(\sqrt{\frac{\pi}{4-\pi}}+E\bar{\lambda}\Big)=-\underline{\mu}\sqrt{V\bar{\lambda}}\Big/\Big(\sqrt{\frac{\pi}{4-\pi}}+E\bar{\lambda}\Big)\\ &={}_f\delta(\infty)-\{{}_f\delta(\infty)-{}_f\delta(0)\}\exp(-{}_f\gamma(\omega, h, \tau_{max})h)\end{aligned} \tag{315}$$

where

$$0<{}_f\delta(\infty)<{}_f\delta(0) \tag{316}$$

substituting eq. (315) in eqs. (160) and (161) the probable expressions of the upper and lower limits of velocity response spectra of the quasi-stationary random excitations are obtained as follows:

$$\sup_E \atop \inf_E\; S_V=(1\pm{}_f\delta(\omega, h, \tau_{max}))\Big(\frac{\sqrt{\pi}}{2}+\sqrt{1-\frac{\pi}{4}}\,{}_f\lambda(\omega, h, \tau_{max})\Big)\sqrt{\sup_\tau S_{E\xi}} \tag{317}$$

In usual cases, ${}_f\delta(\omega, h, \tau_{max})$ given by eq. (315) seems to be a weak function of ω and τ_{max} for all ω and τ_{max}. Consequently, from the practical point of view this function may be replaced by a function of h only, ${}_f\delta(h)$. From this aspect, by assuming that the normalized random variable of the response spectrum is independent of both parameters ω and τ_d, its probability distribution is evaluated as a function of h based upon the results of numerical analyses of response spectra of the linear systems having various frequency parameters subjected to the quasi-stationary random excitations which were previously described by $a(\tau)=1$, $\tau_d=30$ sec and eqs. (291)$\sim$(293). By making use of the above-obtained probability distribution associated with the velocity response spectrum and by taking into consideration eqs. (162), (163) and (315) the function ${}_f\delta(h)$ is approximately evaluated as follows:

$$\begin{aligned} {}_f\delta(h) &= 0.45[1+\exp\{-{}_f\gamma(h)h\}] \\ {}_f\gamma(h) &= 30+80\exp(-40h) \end{aligned} \tag{318}$$

7. Concluding Remarks

As a basic study to obtain a reasonable statistical model of earthquake excitations in the dynamic response analysis of structures, the relation between the quasi-stationary random excitations and their response spectra is discussed. By supposing that the maximum value of the output response of a linear oscillator subjected to an arbitrary excitation is approximately equal to the maximum value of envelope of the output response, the mean value and the upper and lower limits of response spectra of the quasi-stationary random excitations having a finite amplitude probability distribution and a finite duration time are considered, based on a semi-analytical method.

The mean value and the upper and lower limits of response spectra of such quasi-stationary random excitations are expressed by the product of the maximum value of the root mean square of the envelopes of the random output responses, which can be determined analytically as the root of the maximum value of the energy spectral density associated with a modified quasi-stationary random excitations, and the relevant multiplication factors which are approximately expressed in terms of the maximum value, the mean value and the standard deviation of the maximum normalized random variable associated with the pseudo-stationary random process having the ergodic properties and the Rayleigh distribution and are determined semi-experimentally by means of simulation techniques. Of course, the exact analytical approach to this problem is to find the probability distribution of the maximum value of random output response of a linear oscillator. However, since such an analytical approach may be very difficult except for the extremely simple input excitations, the semi-analytical method which is applicable to a general class of quasi-stationary random excitations with an arbitrary envelope and power spectral density is adopted in this paper.

The quasi-stationary random excitations considered here are supposed to be expressed by the product of a cutoff operator which concerns the duration time of excitations, a deterministic continuous function of time which gives the averaged envelope of random excitations and an ergodic stationary Gaussian random process.

The analytical expressions of the energy and power spectral densities of the modified quasi-stationary random process are obtained in the case of the quasi-stationary random excitations having an arbitrary deterministic envelope and the power spectral density of the stationary random process, which is expressed as the product of the band-limiting operator and a rational function of the frequency. The successive approximation procedure of determining the maximum value of the energy spectral density of the modified quasi-stationary random process is also discussed. On the other hand, the probable expressions of the multiplication factors which give the mean value and the upper and lower limits of response spectra of the quasi-stationary random excitations together with the maximum value of the energy spectral density of the modified quasi-stationary random process are determined by means of the simulation technique for the case where the envelope is a slowly varying time-function and the power spectral density of the stationary process is sufficiently flat over the wide frequency range.

From the aspect of obtaining a reasonable model of earthquake excitations for the dynamic response analyses of structures the study made in this paper is only one of the basic studies required for this purpose, hence many related problems remain for future studies. For instance, there may be problems of how to obtain information about the seismicity and dynamic characteristics of the ground at the site of a structure and how to apply this information to the supposition of a model of earthquake excitations in the dynamic response analysis of the structure. There may also be the problem of what modifications should be made for a model of earthquake excitations depending upon the expected dynamic characteristics and measures of the aseismic safety of the structure in connexion with the substantially indeterminate character of earthquakes which will occur in the future. Also, in relation to the coupling phenomenon between structure and soil ground, the problems of where the input excitations should be given and of what modifications should be made for a model of earthquake excitations depending on the point of input excitations and the dynamic characteristics of soil ground and structure may be important, particularly for structures on soft clay or loose sand. As a rule, the supposition of a model of earthquake excitations is not independent of the supposition of a dynamic model of the ground-structural system. It seems that the problem of coupling is to be solved by a reasonable supposition of a dynamic model of the ground-structural system and that the earthquake excitations are to be given at a point outside the coupling region.

In order to develop the study made in this paper in the direction of obtaining a pertinent model of earthquake excitations the envelope and the power spectral density of the quasi-stationary random excitations which are prescribed in rather general forms in this paper are to be reasonably determined according to the seismicity and the dynamic characteristics of the ground at the site of a structure. In this connexion, the random excitations of the stratified visco-elastic medium seem to be one of the most important problems related to the definite supposition of a quasi-stationary random process as a statistical model of earthquake excitations.

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